Self Organized Criticality

Self-organized critical phenomena is exhibited by driven systems which reach a critical state without the tuning of any control parameter

The signature of SOC phenomena (power law distributions, fractality) are ubiquitous in Nature

Critical state is a term common in thermodynamics

A critical point is a point at which a system radically changes its behavior or structure

For instance: phase transitions

(example, the pressure-temperature phase space of a liquid-solid system)

In standard critical phenomena, there is a control parameter which an experimenter can vary to obtain this radical change in behavior

In the case of melting, the control parameter is temperature

Critical states of a system are signaled by a power-law distribution in some observable

Two widely occuring phenomena in nature

- Spatial Fractality: self similarity in space
- $1/f$ noise : Low Frequency Noise

Occurs spontaneously

Evidence of the lack of a natural scale :

Existence of Power Laws

Wide range of natural length and time scales play a role

It is observed that many functional dependences, arising in a variety of contexts, look like straight lines when plotted on a log-log scale

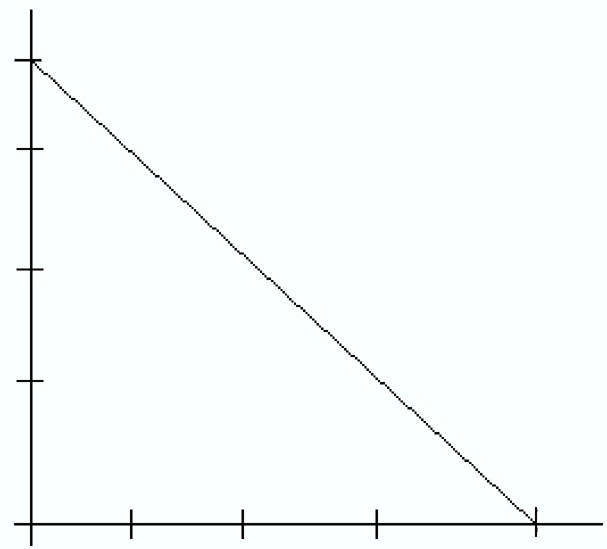
So these functions do not have natural length/time scales :

They are scale invariant or self similar

Power Law: graphed

$N(s) = s^{-t}$ i)

$\log N(s) = -t \log s$ ii)

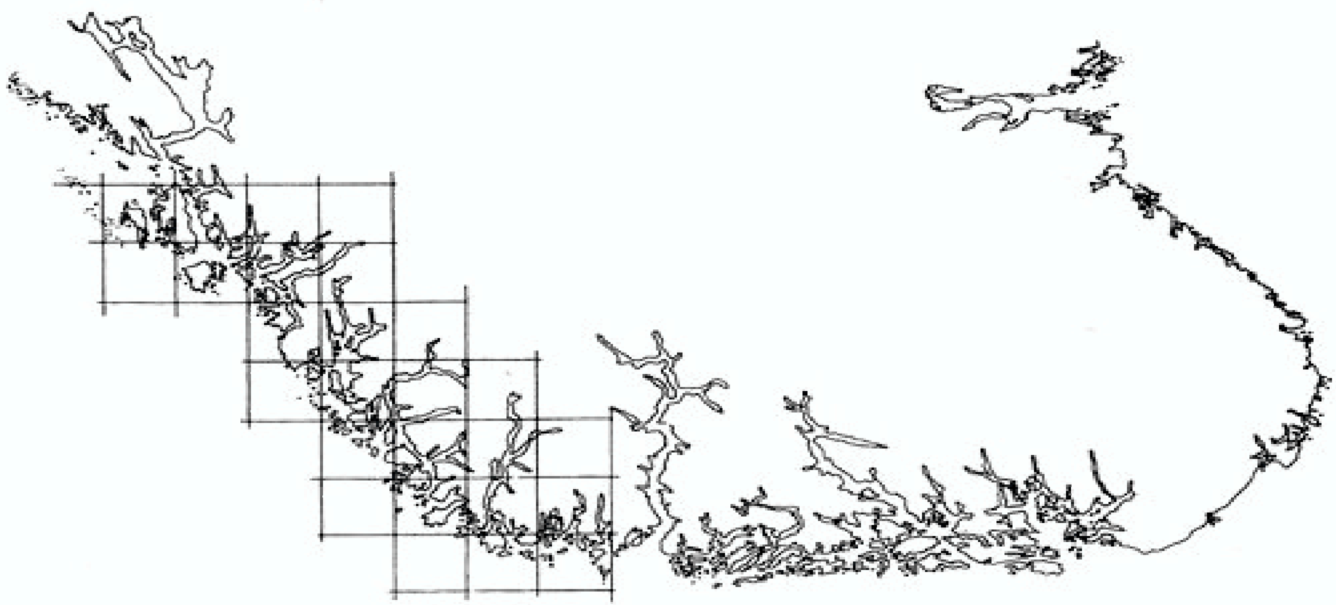


Spatial fractality is ubiquitous in Nature

Present in spatial structures as diverse as :

- Mountain Ranges
- Coastlines
- Clouds
- Colloidal Aggregates
- Patterns of fracture
- Dielectric Breakdown
- Porosity of Soil
- Branching of roots

Fractal: Coast of Norway



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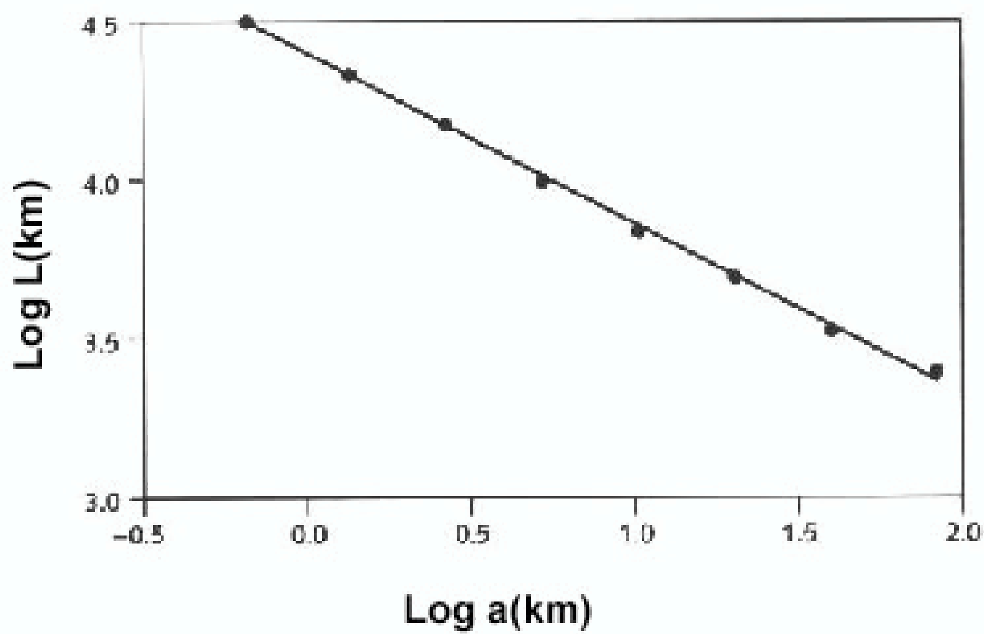
Log (Length) Vs. Log (box size)



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“Fractality” in the temporal regime: $1/f$ noise

$$S(f) \sim \frac{1}{f}$$

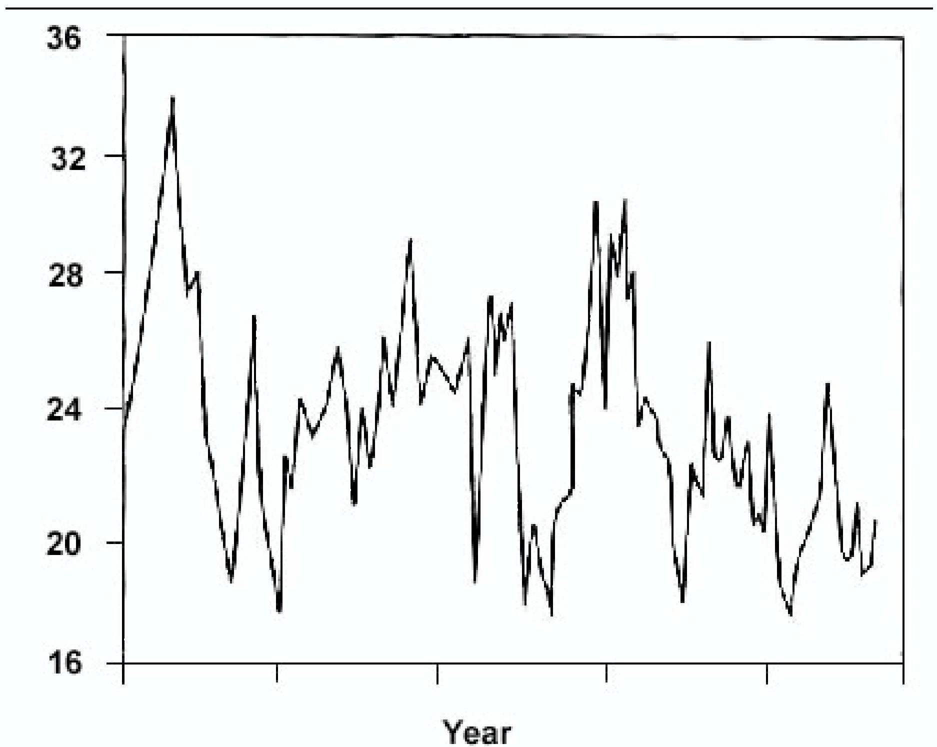
where f is the frequency and $S(f)$ is the spectral power

Also known as flicker noise in astronomy:

in the context of light from quasars; sunspots

Such “noise” (or from some viewpoint “signal”) is characterized by variations on all timescales

1/F Noise



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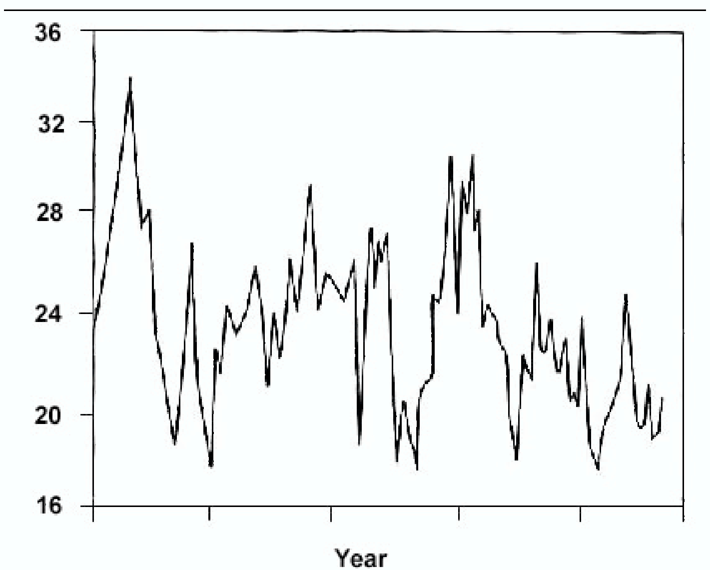
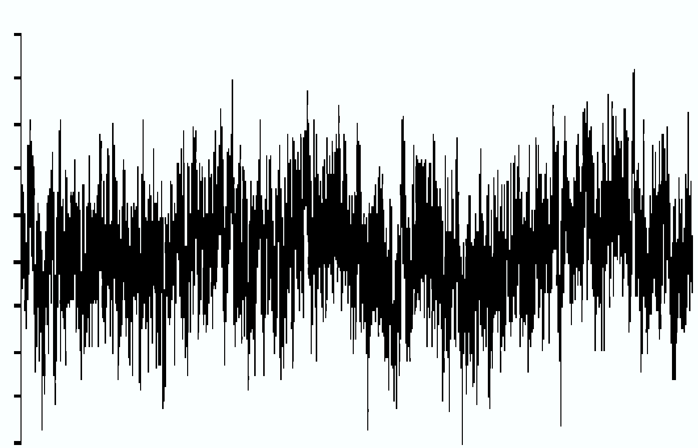
1/f noise has interesting patterns





1/f Noise

White Noise

It is quite unlike white noise

$$S(f) \sim 1/f^0$$

- Equal Power in every unit of bandwidth
- Infinitely choppy or discontinuous everywhere

White noise has a well defined mean : the value converges as we average over longer and longer times

But its instantaneous value is undefined

For White Noise spectrum $S(f) \sim 1/f^0$:

- We can integrate the power from some finite frequency down to zero
- But will meet divergences as we integrate up from a finite frequency to infinity

The $1/f$ noise is also unlike $1/f^2$ noise

$$S(f) \sim 1/f^2$$

Integral of white noise : can be generated for instance by random walk noise

That is a running sum incremented by random steps

The $1/f^2$ noise is very smooth

Has as a well defined value at every point

But its mean is not well defined: as the value of the function wanders further and further away from its intial value

That is, the integral of power down to zero is divergent

Now $1/f$ noise is somehow in between and more interesting than either $1/f^0$ or $1/f^2$ noise

- It is neither too rough nor too smooth
- It has rapid fluctuations down to numerical resolution limits
- Also posseses strikingly global trends :

  which the eye may be tempted to interpret as “cyclic”

- It is divergent when integrated either to zero or to infinity
- It has neither a well defined mean nor a well defined instantaneous value
- But these divergences are logarithmic, and thus slow enough, so that even to very high or low cutoffs the appearance of the noise hardly changes
- Hence its scale invariance or self similarity

Examples of low frequency noise are widespread

It appears in events as varied as:

- Resistance fluctuations
- Sand flow in an hourglass
- Solar Flares
- Stock Market Fluctuations (so says Per Bak)

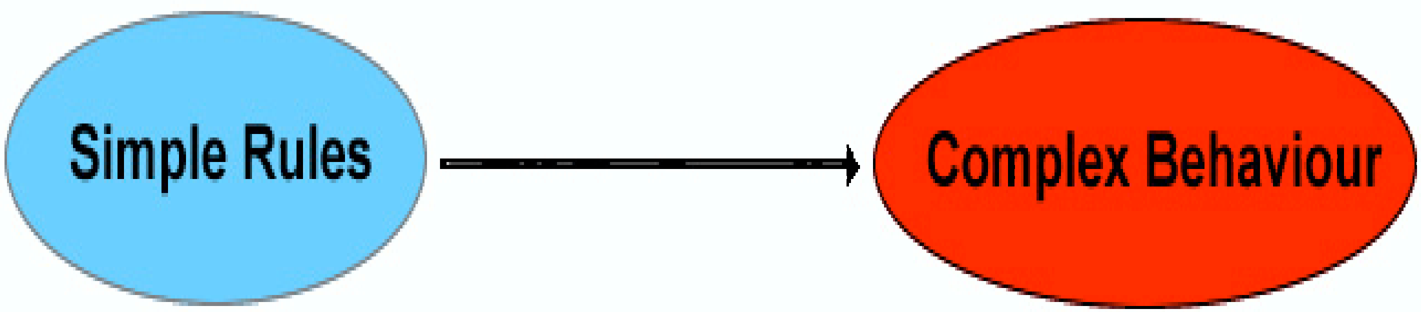
The generality and ubiquity of these two phenomena have puzzled scientists for a long time

The study of fractality is often confined to the level of characterization (for example plotting results of experiments on log-log scales and associating the slope with some kind of fractal dimension)

What theorists would really like to do is to find the dynamical origin of these phenomena

Construct models that yield space-time fractality from very simple local rules and interactions

Emergent Complexity



Self-organized critical phenomena, by contrast to (thermodynamic) crtical phenomena, is exhibited by driven systems which reach a critical state by their intrinsic dynamics, independent of the value of any control parameter

In SOC the system self-organizes to the critical point

Quite unlike, say, a system’s pressure and temperature being carefully tuned so that it was exactly at the critical point

Spontaneously yields space-time fractality

The sand pile model is a cellular automata model which allows an intuitive understanding of the principles of the theory of self organized criticality

The underlying algorithm for sandpile models is simple

In essence, the algorithm keeps track of numbers associated with points on a grid

Numbers on the grid can increase, decrease or stay the same

If a number on the grid gets too big then the algorithm decreases that number, and subsequently increases numbers elsewhere

Consider a two-dimensional grid

Each point (x, y) in the grid has an integer $z(x, y)$ associated with it

This number is a counter for the point (x, y) (say running from 0 to 4)

In terms of the sand pile, it can be thought of as the average slope of the sand pile at that point (on the grid)

External Driving

In the course of running the program, the number at a random point within the grid is increased by one unit:

$$z(x, y) \rightarrow z(x, y) + 1$$

In other words, the average slope of the sand pile at that point is increased by one unit

After the perturbation if $z(x, y)$ becomes larger than a threshold : triggers an avalanche

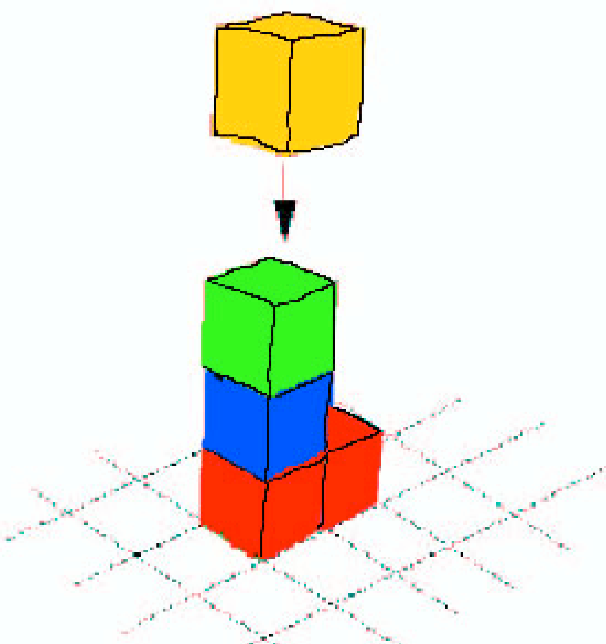
If $z(x,y) > z^*$ (e.g. threshold $z^* = 3$)

Re-distribution (Spill-over; Toppling) occurs :

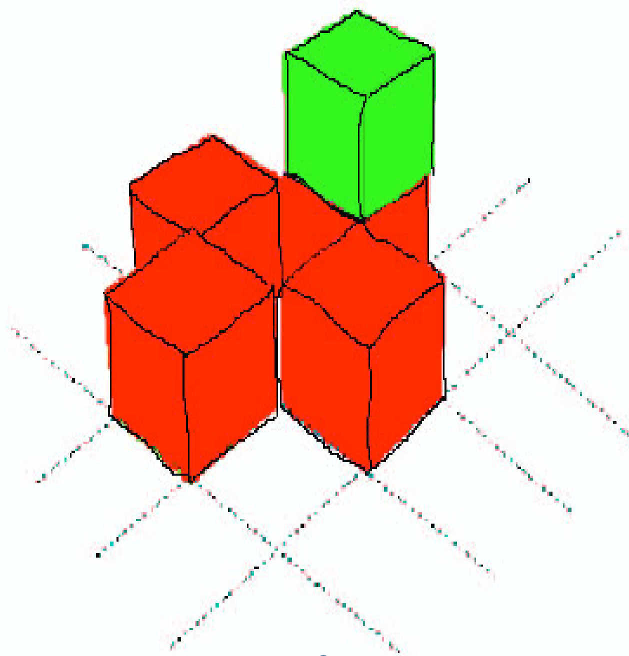
$$z(x,y) \to z(x,y) - 4$$
$$z(x \pm 1, y) \to z(x \pm 1, y) + 1$$
$$z(x, y \pm 1) \to z(x, y \pm 1) + 1$$

In other words, the slope of the sand pile at that point becomes too steep and grains of sand roll down the sand pile to nearby grid points

Before and After



Before



After

If this redistribution causes z to be too big at one of the nearby grid points : then the process continues

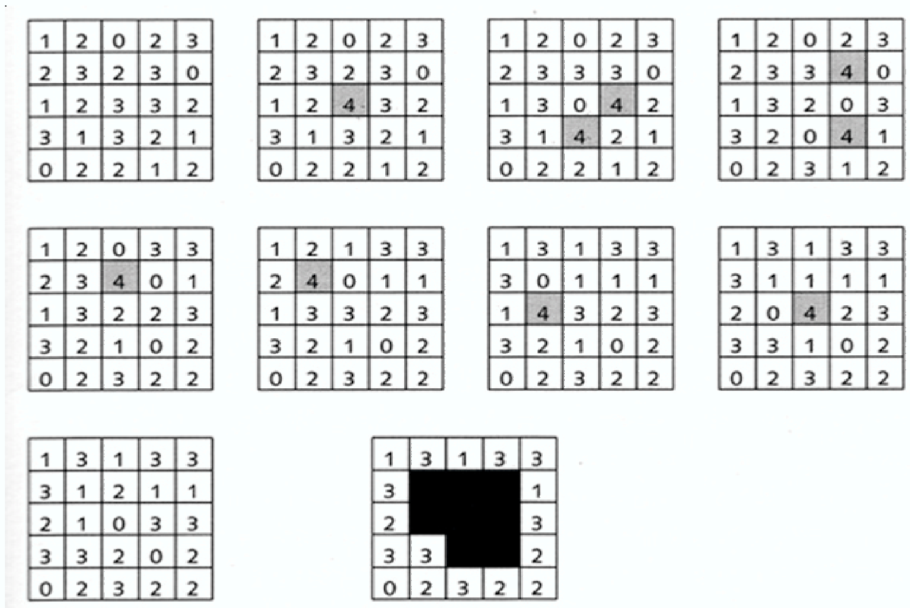
In other words, if the sand rolls to a point that was already poised for an avalanche then the avalanche continues

If the redistribution does not cause z to be too big at one of the nearby points : then the process stops

Model Rules

- Drop a single grain of sand at a random location on the grid
  - Random (x,y)
  - Update model at that point: Z(x,y) → Z(x,y)+1
- If Z(x,y) > Threshold, spark an avalanche
  - Threshold = 3

Example: Domino Effect



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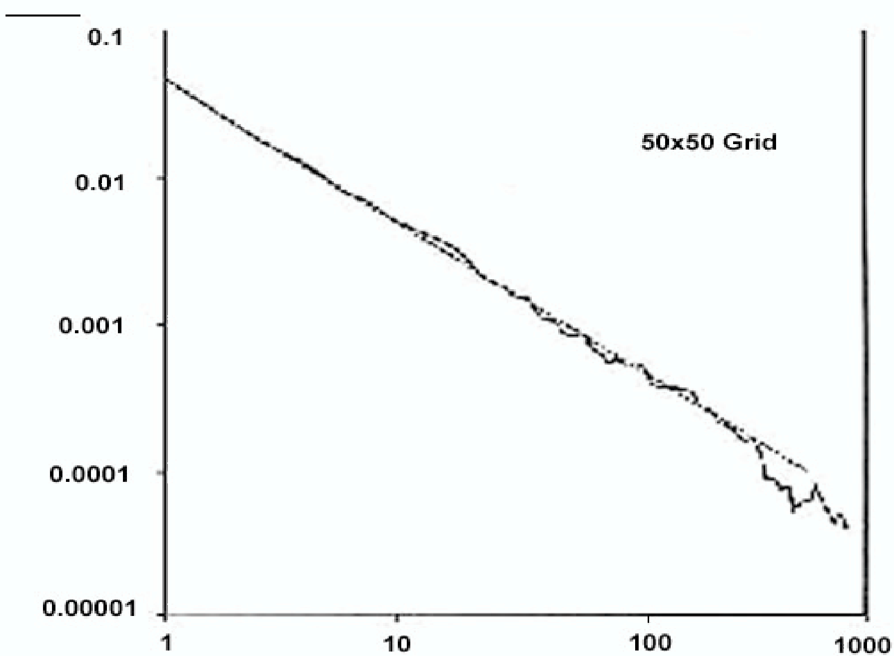
One can imagine that for each point in the grid there is a domain D, which is the set of all points whose state was changed as a result of the perturbation

Each domain D has a finite size s, which is the aerial extent of an avalanche

D as a function of s : clearly shows a power law distribution

Implying the system has self-organized to a critical state

Size Distribution of Avalanches



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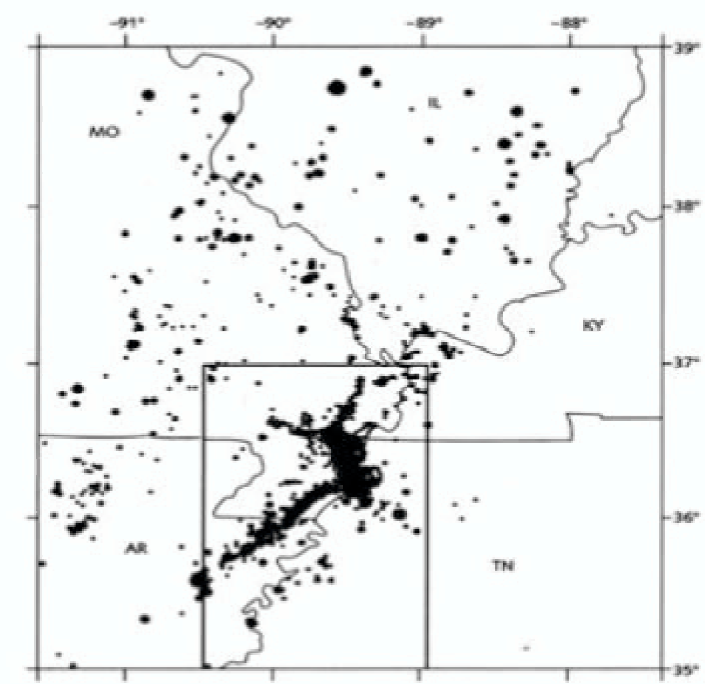
Earthquake distribution



1974-1983

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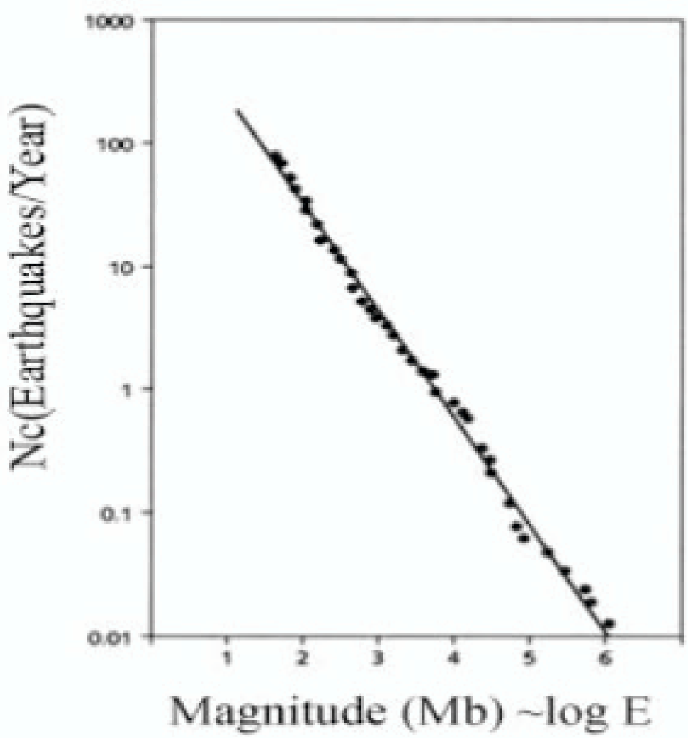
Gutenberg-Richter Law



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The self organized state is Metastable

Non-Equilibrium Steady State

Edge of Chaos

As more grains are added the slope of the pile increases

Eventually, the slope locally reaches a critical value such that the addition of one more grain results in an avalanche

With the addition of still more grains the sandbox will overflow: Sand is thus added and lost from the system

When the count of grains added equals the count of grains lost (on average) then the sand pile has self-organized to a nonequilibrium steady state

- The next avalanche can be of any size:

  ranging from a single grain to a catastrophic collapse of the sand pile

- The size distribution of the avalanches will follow a power law

  e.g. if one were to count the size of avalanches over some period, one would most likely find that there was 1 avalanche of size 1,000, 10 avalanches of size 100, 100 avalanches of size 10, and so on

- Simple physical laws dictate the interactions of individual components: The specifics of these laws are not important, however, as the system will robustly self-organize to a critical state for a variety of laws

  Highly specific physical laws are not necessary for the generation of the power law distribution

- Power Law distributions generated for a variety of conditions

- The surface of the sand pile will have a fractal dimension

The theory of self-organized criticality (SOC) seeks to explain how the multitude of large interactive systems observed in nature develop power law relationships from simple rules of interaction

The most commonly used paradigm of, and perhaps the best way to understand the theory is the sand pile

Their macroscopic behaviour displays the spatial and/or temporal scale-invariant characteristic of the critical point of a phase transition, but, unlike the latter, in SOC these features result without needing to tune control parameters to precise values