Models of Spatiotemporal Phenomena

Two ingredients:

- Local Dynamics :
  gives change of state locally (at a site/node)
  Source of local temporal patterns

- Interactions :
  Spatial transmission of energy and information
  Source of global spatial patterns

Broad classification of models can be obtained by different possible discretization schemes

Model	Time	Space	Variables
Partial Differential Equations	C	C	C
Oscillator Chains	C	D	C
Lattice Dynamical Systems	D	D	C
Cellular Automata	D	D	D

State variable(s) corresponds to the physical variable of interest

e.g. temperature, pressure, velocity field, concentrations, populations, electric fields

Choice of local dynamics

Typical examples:

- Logistic Maps $f(x) = rx(1-x)$
- Circle Maps $f(x) = \omega + x + k\sin(2\pi x)$
- Piecewise linear functions;
  Such as tent maps or shift maps
- Higher-dimensional maps (eg. Henon map)

Choice of Coupling Classes

- Local : couples with neighbours
- Global : couples with all (mean-field type)
- Non-local : can couple (randomly) with sites far-away
  Allows short-cuts in spatial conections
- Homogeneous (independent on site/node)
- Heterogeneous (varies with site/node)

Boundary Conditions

- Fixed
- Periodic
- Free
- Noise driven
- Periodically driven

Initial Conditions

For instance:

(i) Random distribution

Selects out a statistically probable pattern

generic behaviour

Problem: does not admit a natural continuum limit

(ii) Spatially periodic

Characterization

Visualization of dynamical phases

- Space-time plots
- Time evolution of a generic site
- Bifurcation Diagram

Quantitative Measures

- Fourier Transforms (Space and Time)
- Eigenvalue Spectrum; N Lyapunov Exponents

Lattices to Networks

Spatiotemporal Consequences of Random Nonlocal Coupling

Synchronous Evolution ("Global Clock")
vis-a-vis
Random Updating

Strong reasons to re-visit the fundamental issue of interactions defined on a regular lattice-like structure in Coupled Map Lattices

Some degree of randomness in spatial coupling can be closer to physical reality than strict nearest neighbour scenarios

Many systems of biological, technological and physical significance are better described by randomising some fraction of the regular links

What are the spatiotemporal consequences of rewiring some of the coupling connections randomly

What happens to the dynamics of an extended system comprised of a collection of elemental dynamical units with varying degrees of randomness in its spatial connections

Consider a one-dimensional ring of coupled logistic maps

The sites (nodes) are denoted by integers $i = 1, \ldots, N$ where N is the linear size of the lattice

On each site is defined a continuous state variable denoted by $x_n(i)$

Corresponds to the physical variable of interest

The evolution of this lattice in discrete time n under standard nearest neighbour interactions :

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{2}\{x_n(i+1) + x_n(i-1)\}$$

Strength of coupling : ϵ

The local on-site map is chosen to be the fully chaotic logistic map:

$$f(x) = 4x(1-x)$$

Now consider the system with its coupling connections rewired randomly in varying degrees dynamically

At every update we will connect a fraction p of randomly chosen sites in the lattice to two random sites

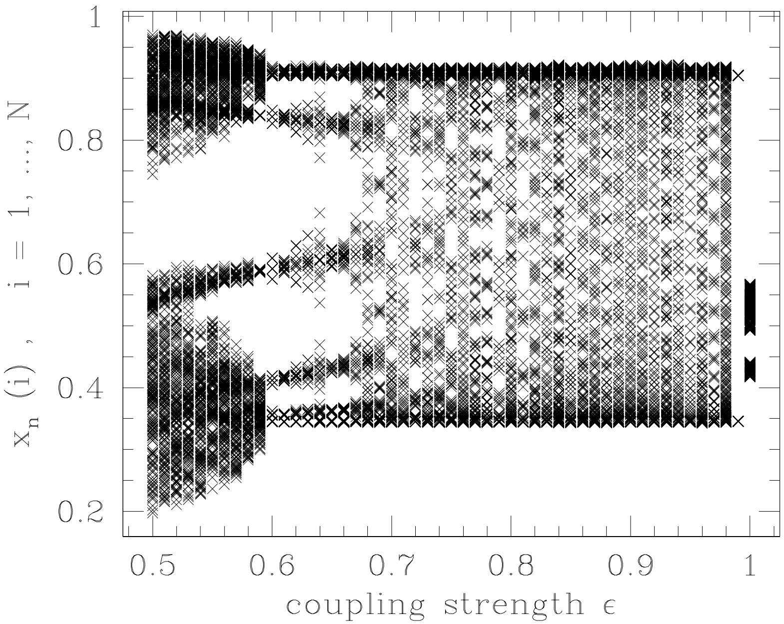
That is, we will replace a fraction p of nearest neighbour links by random connections

$p = 0$: corresponds to the usual nearest neighbour interaction

$p = 1$: corresponds to completely random coupling

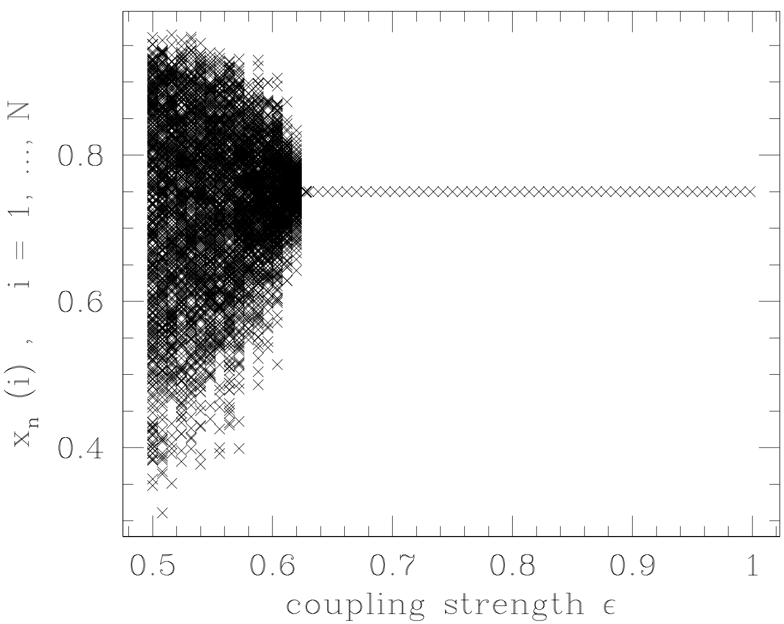
Explore the full range of $p \quad (0 \leq p \leq 1)$

Try to determine what dynamical properties are significantly affected by the way connections are made between elements



Coupled logistic maps with regular nearest neighbour connections

Bifurcation diagram clearly shows that the standard nearest neighbour coupling does not yield a spatiotemporal fixed point anywhere in the entire coupling range $0 \leq \epsilon \leq 1$



Coupled logistic maps with completely random connections

Creates windows in parameter space where a spatiotemporal fixed point state gains stability

Onset of spatiotemporal fixed point : ϵ_{bifr}

For completely random coupling $p = 1$: $\epsilon_{bifr} \sim 0.62$

For all $p > 0$: there is a stable region of synchronized fixed points

In the stable region of synchronized fixed points

namely, in the parameter interval $\epsilon_{bifr} \leq \epsilon \leq 1.0$:

All lattice sites i are synchronized at $x_n(i) = x^* = 3/4$

Where $x^* = f(x^*)$ is the fixed point solution of the individual chaotic maps

x^* : strongly unstable in the local chaotic map

Analyse this system to account for the much enhanced stability of the homogeneous phase under random connections

Only possible solution for a spatiotemporally synchronized state :

All $x_n(i) = x^*$ only when $x^* = f(x^*)$

For the case of the logistic map at $r = 4$:

Fixed point solution of the local map $x^* = 4x^*(1 - x^*) = 3/4$

To calculate the stability of the lattice with all sites at x^* we will construct an average probabilistic evolution rule for the sites :

mean field version of the dynamics

Some effects due to fluctuations are lost, but as a first approximation we have found this approach qualitatively right, and quantitatively close to to the numerical results as well

All sites have probability p of being coupled to random sites, and probability $(1-p)$ of being wired to nearest neighbours

Then the averaged evolution equation of a site j is

$$x_{n+1}(j) = (1-\epsilon)f(x_n(j)) + (1-p)\frac{\epsilon}{2}\left\{x_n(j+1) + x_n(j-1)\right\}$$
$$+p\frac{\epsilon}{2}\left\{x_n(\xi) + x_n(\eta)\right\}$$

where ξ and η are random integers between 1 and N

Linear Stability Analysis of the coherent state:

Replacing $x_n(j) = x^* + h_n(j)$, and expanding to first order gives

$$h_{n+1}(j) = (1-\epsilon)f'(x^*)h_n(j) + (1-p)\frac{\epsilon}{2}\left\{h_n(j+1) + h_n(j-1)\right\}$$
$$+p\frac{\epsilon}{2}\left\{h_n(\xi) + h_n(\eta)\right\}$$

$$\approx (1-\epsilon)f'(x^*)h_n(j) + (1-p)\frac{\epsilon}{2}\left\{h_n(j+1) + h_n(j-1)\right\}$$

i.e., to a first approximation one can consider the sum over the fluctuations of the random neighbours to be zero

This approximation is clearly more valid for small p

For stability considerations one can diagonalize the above expression using a Fourier transform

$$h_n(j) = \sum_q \phi_n(q) \exp(ijq)$$

where q is the wavenumber and j is the site index

This finally leads us to the following growth equation:

$$\frac{\phi_{n+1}}{\phi_n} = f'(x^*)(1-\epsilon) + \epsilon(1-p)\cos q$$

with q going from 0 to π

Clearly the stabilization condition will depend on the nature of the local map $f(x)$ through the term $f'(x)$

Considering the fully chaotic logistic map with $f'(x^*) = -2$, one finds that the growth coefficient that appears in this formula is smaller than one in magnitude if and only if

$$\frac{1}{1+p} < \epsilon < 1$$

i.e.

$$\epsilon_{bifr} = \frac{1}{1+p}$$

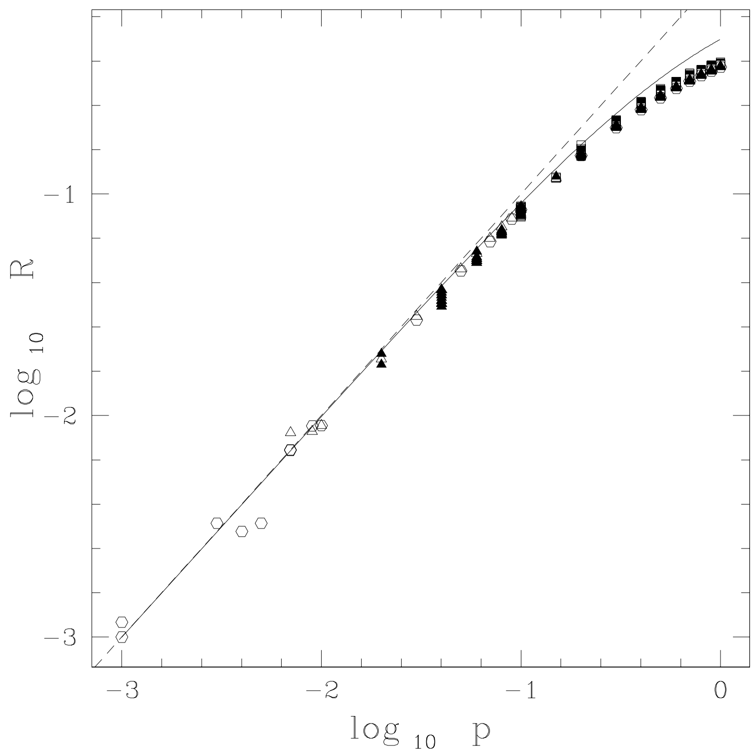
The range of stability $\mathcal{R} = 1 - \epsilon_{bifr}$ is

$$\mathcal{R} = 1 - \frac{1}{1+p} = \frac{p}{1+p}$$

For small p ($p << 1$) standard expansion gives

$$\mathcal{R} \sim p$$

- Regular nearest neighbour couplings ($p = 0$) gives a null range for spatiotemporal regularity
- Fully random connections ($p = 1$) yields the largest stable range



The stable range $\mathcal{R}$, within which spatiotemporal homogeniety is obtained, with respect to the fraction of randomly rewired sites p

The different points are obtained from numerical simulations over several different initial conditions, for 4 different lattice sizes, namely $N = 10, 50, 100$ and 500

Lattice size has very little effect on this synchronization

The solid line displays the analytical result

It is clear that for a large range of p there is very good agreement with the analytical formula

Clearly evident : random coupling leads to large parameter regimes of regular homogeneous behaviour, with all lattice sites synchronized exactly at x^*

The synchronized spatiotemporal fixed point gains stability over some finite parameter range for finite p,

Namely, whenever $p > 0$, however small, we have $\mathcal{R} > 0$

So any degree of randomness in spatial coupling connections opens up a synchronized fixed point window

Results from Other Models

- Coupled tent maps, with the local map given as:

$$f(x) = 1 - 2|x - 1/2|$$

Unstable fixed point : $x^* = 2/3$

Local slope : $f'(x^*) = -2$

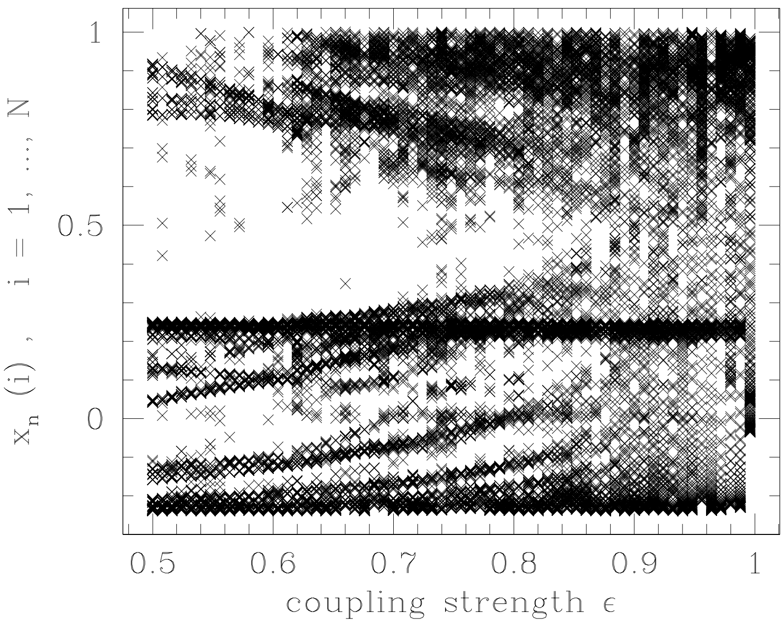
- Coupled circle map networks, where the local map is

$$f = x + \Omega - \frac{K}{2\pi} \sin(2\pi x)$$

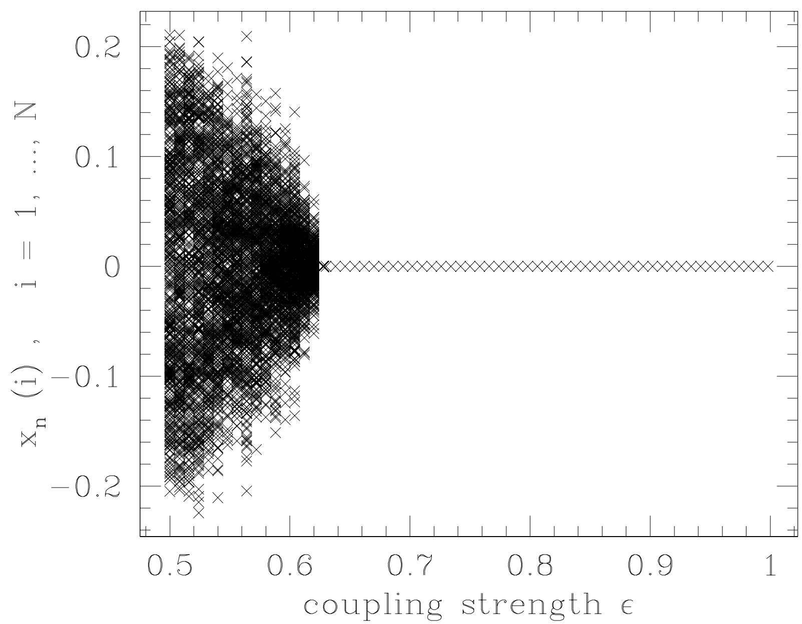
A representative example: $\Omega = 0, K = 3$
Unstable fixed point : $x^* = \frac{1}{2\pi} \sin^{-1}(\Omega/K)$

Local slope : $f'(x^*) = -2$

In both systems random rewiring yields stable spatiotemporal synchronisation



Coupled sine circle maps with strictly regular nearest neighbour connections

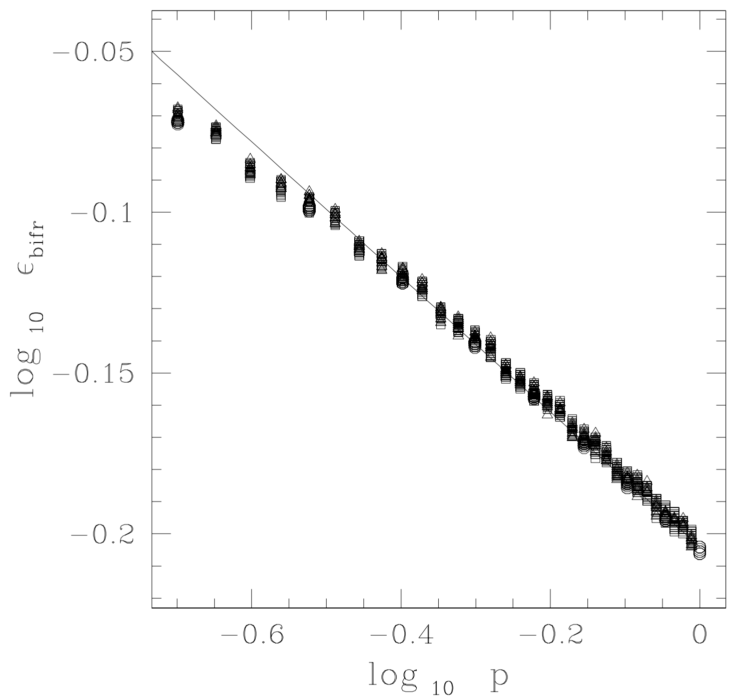


Coupled sine circle maps with completely random connections

- Since $f'(x^*) = -2$ for both the tent map and the circle map, we expect from our analysis that their ϵ_{bifr} and $\mathcal{R}$ will be the same as for the logistic map

- This agrees with simulations :

  Numerically obtained ϵ_{bifr} values for ensembles of coupled tent, circle and logistic maps fall indistinguishably around each other, even for high p where the analysis is expected to be less accurate



Coupled tent maps (open squares)
Coupled circle maps (open triangles)
Coupled logistic maps (open circles)

Outlook

- The regularising effect of random coupling can help in designing efficient control methods for spatially extended interactive systems
- It also suggests a different mechanism for regulation in natural physical and biological systems

The standard time evolution of coupled map lattices employ parallel (or synchronous) updates in which all individual maps of the lattice are iterated forward simultaneously

Asynchronous Updating of Coupled Maps

Updates are not concurrent, but sequential

2-dimensional square lattice

Sites : denoted by two sets of integers i and j,
$i, j = 1, \ldots, N$, where N is the linear size of the lattice

State variable : denoted by $x_n(i, j)$

Local on-site map:

$$f(x) = rx(1 - x)$$

The nonlinearity parameter r is chosed to be 4,
i.e. the local dynamics is completely chaotic

Break the lattice into N_c disjoint randomly chosen subsets

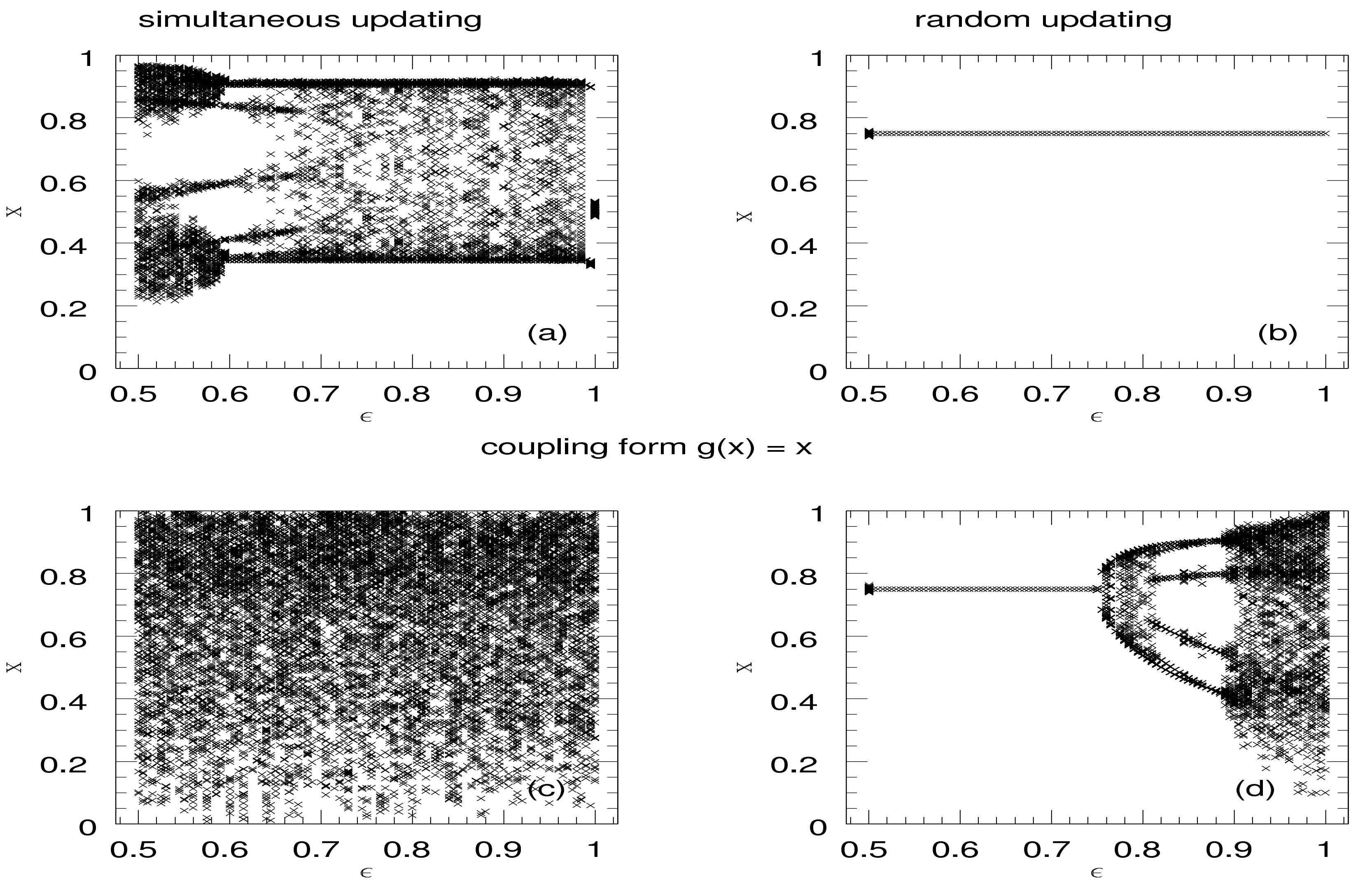
Update the sites belonging to each subset simultaneously, while updating the different subsets sequentially

- $N_c = 1$: Parallel/Synchronous updates
- $N_c = N$: Completely Asynchronous Random update
- Value of N_c indicates varying degrees of asynchronicity in the evolution

For instance, in a lattice of 9 elements (lattice size $= 3 \times 3$), say we wish to update in 3 unequal subsets ($N_c = 3$), with the subsets being of size 3, 4 and 2

Update the lattice between time n and $n+1$ in 3 sequential steps: first we choose at random 3 sites (i, j) and update them, then we update 4 sites randomly chosen from the remaining 6 sites, and finally we update the remaining 2 sites belonging to the last subset

In the subsequent unit of time, from $n+1$ to $n+2$ we again choose at random, first 3 sites and update them, then 4 sites from the remaining sites and update them, followed by the update of the last 2; and so on...



coupling form g(x) = x

coupling form g(x) = f(x)

Dynamics on Lattices/Networks : rich in phenomena

Therefore they have great potential as models

Variations in connectivity and updating rules : dynamical consequences may be profound