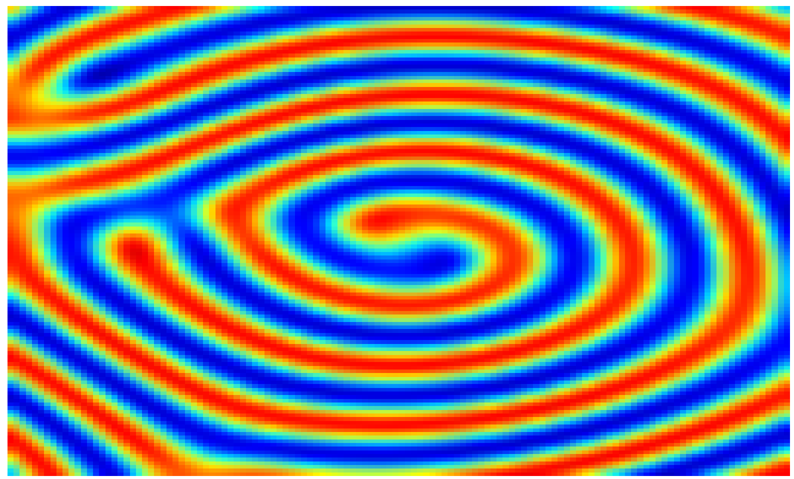
Models of Spatiotemporal Phenomena



Aim:

To model the diversity of pattern generation in spatially extended systems

Provide suggestive conceptual frameworks for understanding complex phenomena which are generic in physical systems

Construct simple yet effective models which are capable of capturing the essence of complicated dynamic processes as they occur in nature

For instance, unimodal maps of an interval on to itself : versatile paradigm for low-dimensional dissipative systems

Though one-dimensional : neither too simple nor too specific

In fact it can yield a rich spectrum of dynamical behaviour

From low-dimensional systems : move towards nonlinear extended systems

Spatial systems composed of a large number of low-dimensional units

Build prototypes that can yield a repertoire of behaviours reminiscent of behaviour widely observed in large interactive systems where spatial extent is crucial

For instance: turbulence, pattern formation, spatio-temporal intermittency, josephson junction arrays, optically bistable devices, neural dynamics

An important prototype of complex systems:

nonlinear dynamical systems with spatially distributed degrees of freedom

Lattices or networks of large number of dynamical systems

Two ingredients:

- Local Dynamics :
  gives change of state locally (at a site/node)
  Source of local temporal patterns
- Interactions :
  Spatial transmission of energy and information : global organizing principles arises from this interconnectivity

For instance

Local dynamics may be a chaotic map :
Creation of local instabilty through the local chaos

Mechanism for transmitting information among nodes :
diffusive; threshold-activated

Functional form of the coupling; Strength of the coupling;
Extent of the coupling

Models of spatiotemporal phenomena

3 quantities are to be accounted for:

- Time
- Space
- State Variable(s)

Broad classification of models can be obtained by different possible discretization schemes

Model	Time	Space	Variables
Partial Differential Equations	C	C	C
Oscillator Chains	C	D	C
Lattice Dynamical Systems	D	D	C
Cellular Automata	D	D	D

Example of Cellular Automata

Sandpiles : model of Self Organized Criticality (SOC)

Examples of Lattice Dynamical Systems

Coupled Map Lattices (CML)

Identify dynamic processes and concepts that can be employed to understand large families of complicated systems

Universality Classes in Pattern Dynamics in Coupled Map Lattices (CML)

Qualitative universality classes

- Holds for a wide range of CMLs
- Found in experiments on physical systems

- Size (number of units) : N

  In the context of 1-d lattices : “length” of chain/ring

  In the context of a network : Number of Nodes

- Discrete time : n

- Spatial index : i (nodes/sites $i = 1, 2 \dots N$)

- State of the system : $x_n(i)$ or x_n^i

  i.e. $x_n(1), x_n(2) \dots x_n(N)$

  Can form a pattern or field

On a spatial lattice a local dynamical variable x_n^i is assigned

The evolution of the local variable x_n^i is governed by

(i) Local Dynamics : f(x)

e.g. $f(x) = rx(1-x)$
where r is the local nonlinearity

(ii) Coupling of strength ϵ

e.g. Nearest Neighbour Coupling, namely site i is influenced by sites $i-1$ and $i+1$

Choice of local dynamics

Typical examples:

- Logistic Maps $f(x) = rx(1-x)$
- Circle Maps $f(x) = \omega + x + k\sin(2\pi x)$
- Piecewise linear functions;
  Such as tent maps or shift maps
- Higher-dimensional maps (eg. Henon map)

Choice of Coupling Classes

- Local (couples with “neighbours”)
- Non-local (can couple with sites “far-away”)

  Allows short-cuts in spatial conections

General nearest neighbour interaction evolution function:

$$x_{n+1}^{i} = f(x_n^i) + \epsilon_0 g(x_n^i) + \epsilon_R g(x_n^{i+1}) + \epsilon_L g(x_n^{i-1})$$

$(\epsilon_0, \epsilon_R, \epsilon_L)$: coupling kernel

$g(x)$ is the coupling dynamics

Linear Coupling : $\quad g(x) = x$

Future Coupled : $\quad g(x) = f(x)$

Homogeneous; Heterogeneous

$$x^i_{n+1} = f(x^i_n) + \epsilon_0 g(x^i_n) + \epsilon_R g(x^{i+1}_n) + \epsilon_L g(x^{i-1}_n)$$

- Additive Coupling : $\epsilon_0 = 0; \epsilon_R = \epsilon_L$
- Laplacian Coupling : $-\frac{\epsilon_0}{2} = \epsilon_R = \epsilon_L$
- Totalistic Coupling : $\epsilon_0 = -\frac{2}{3}; \epsilon_R = \epsilon_L = \frac{1}{3}$
- Unidirectional Coupling : $-\epsilon_0 = \epsilon_L; \epsilon_R = 0$

Asymmetric Coupling : models Open Flows

Boundary Conditions

- Fixed
- Periodic
- Free
- Noise driven
- Periodically driven

Initial Conditions

For instance:

(i) Random distribution

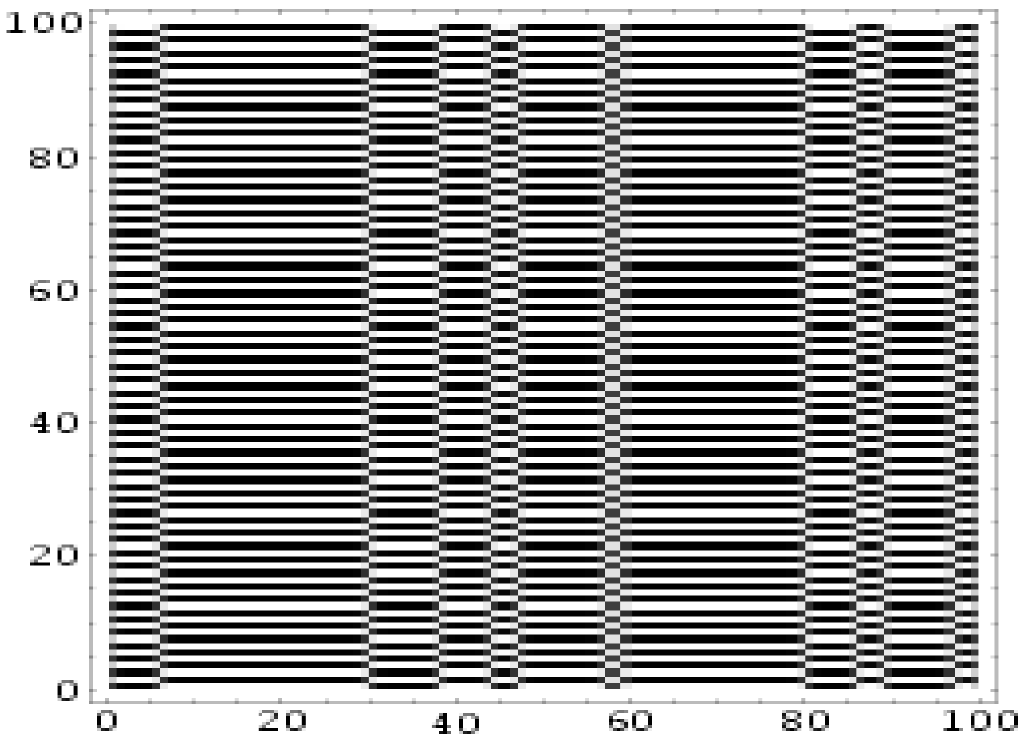
Selects out a statistically probable pattern

generic behaviour

Problem: does not admit a natural continuum limit

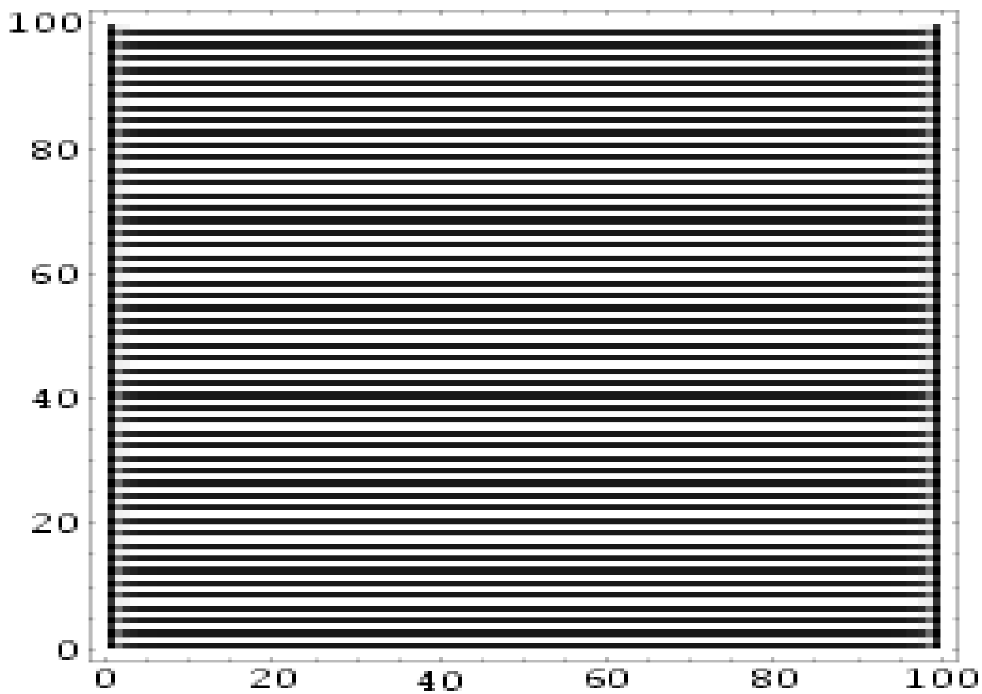
(ii) Spatially periodic

Spatiotemporal periodicity



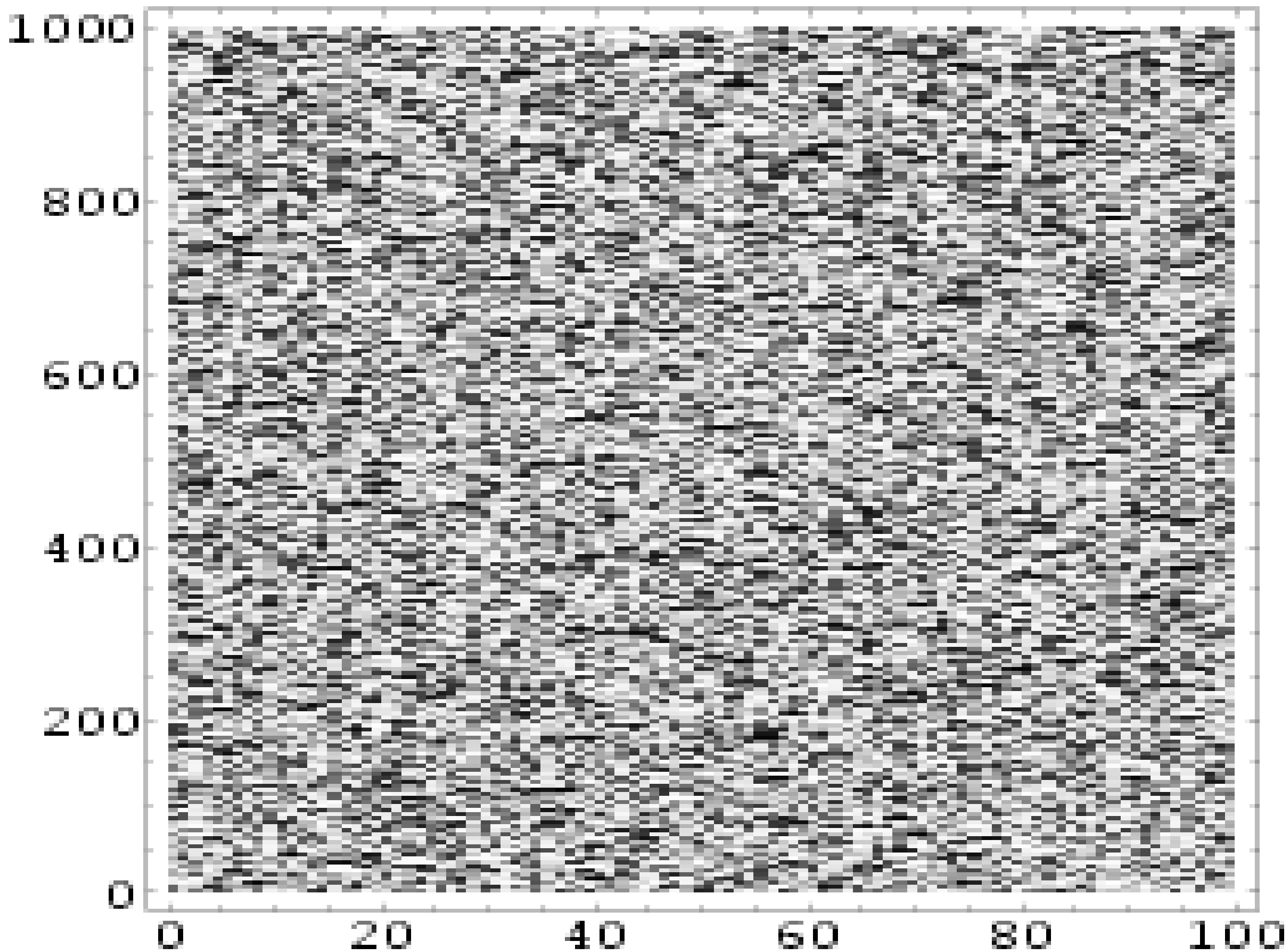
$r = 3.2; \epsilon = 0.1$

Temporal periodicity ; Spatial Coherence

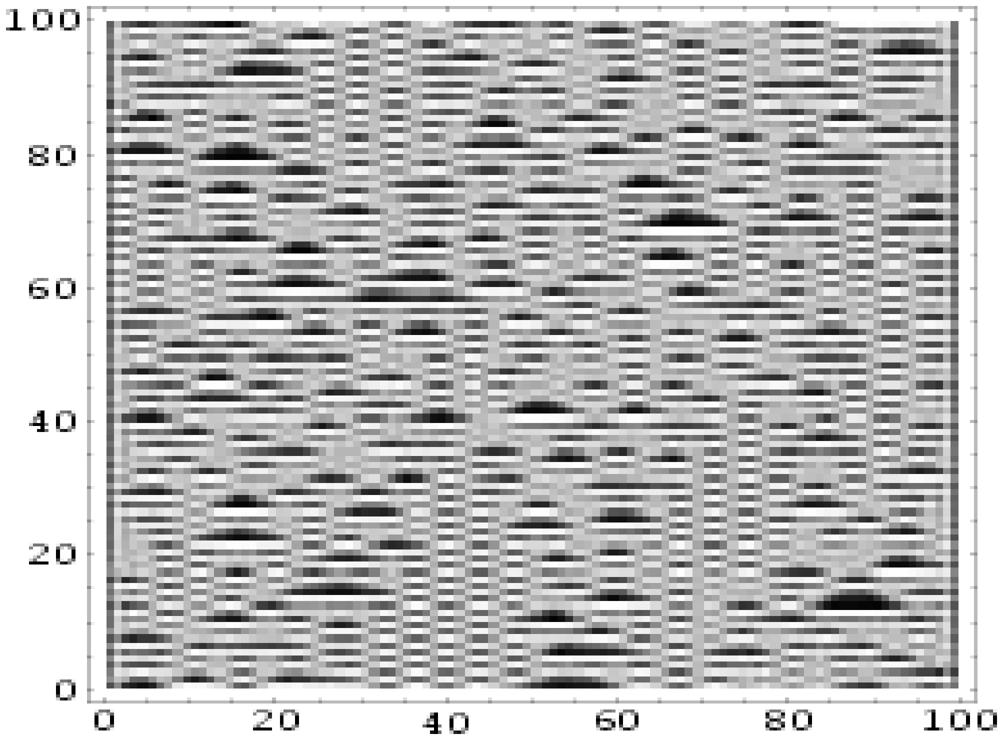


$r = 3.2; \epsilon = 0.6$

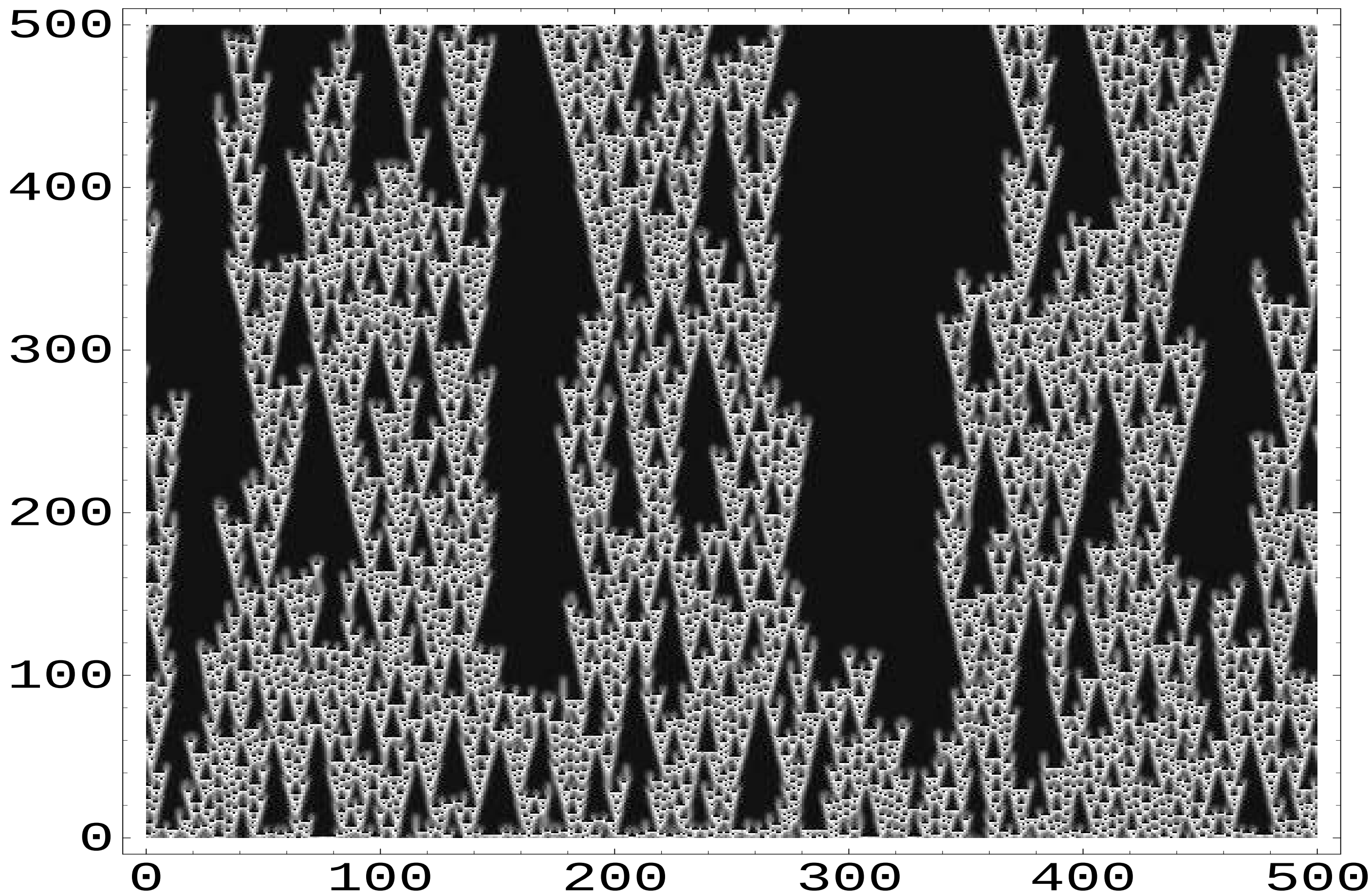
Fully Developed Spatiotemporal Chaos



Spatiotemporal Chaos



$r = 4; \epsilon = 0.8$



Spatio-temporal Intermittency : Class of Directed Percolation

Transition from an ordered pattern to fully developed spatiotemporal chaos occurs via spatiotemporal intermittency

In spatiotemporal intermittency : laminar motion and turbulent bursts in space time

Laminar motion : periodic or weakly chaotic dynamics with spatially regular structure

Turbulent bursts: No spatio-temporally regular structure

In space-time plots : black (bursts) and white (laminar)

2 types of spatio-temporal intermittency

Type I STI : no spontaneous creation of bursts

If a site and its neighbours are laminar : site remains laminar in the next step

Before onset of STI : stable state with spatial homogeneity and temporal periodicity

Dynamics very similar to some probabilistic CA with 2 states corresponding to burst (active) and laminar (inactive/passive)

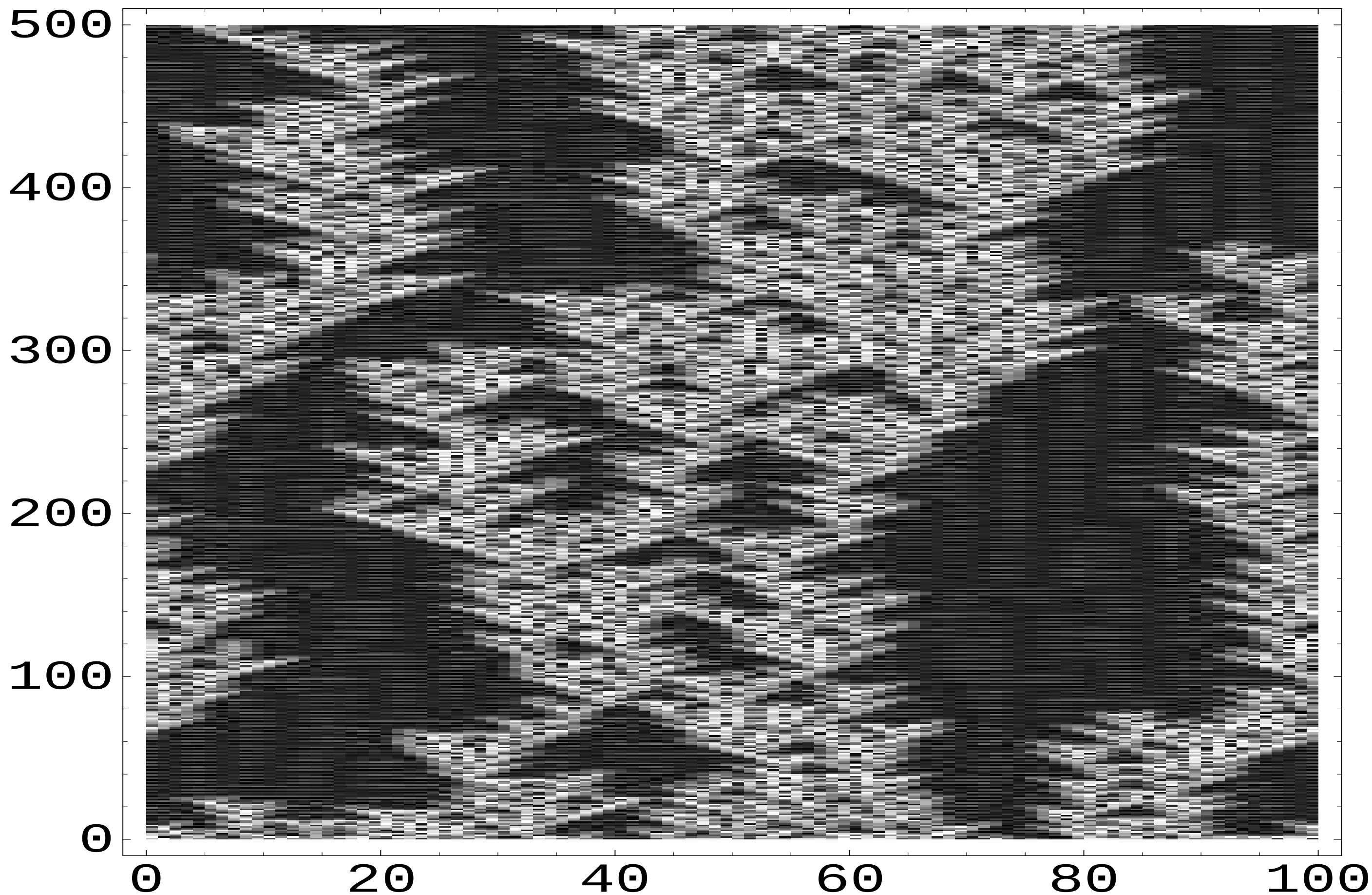
Directed Percolation analogy

Type II STI : Spontaneous creation of turbulent bursts when some coarse grained reduction of states is used

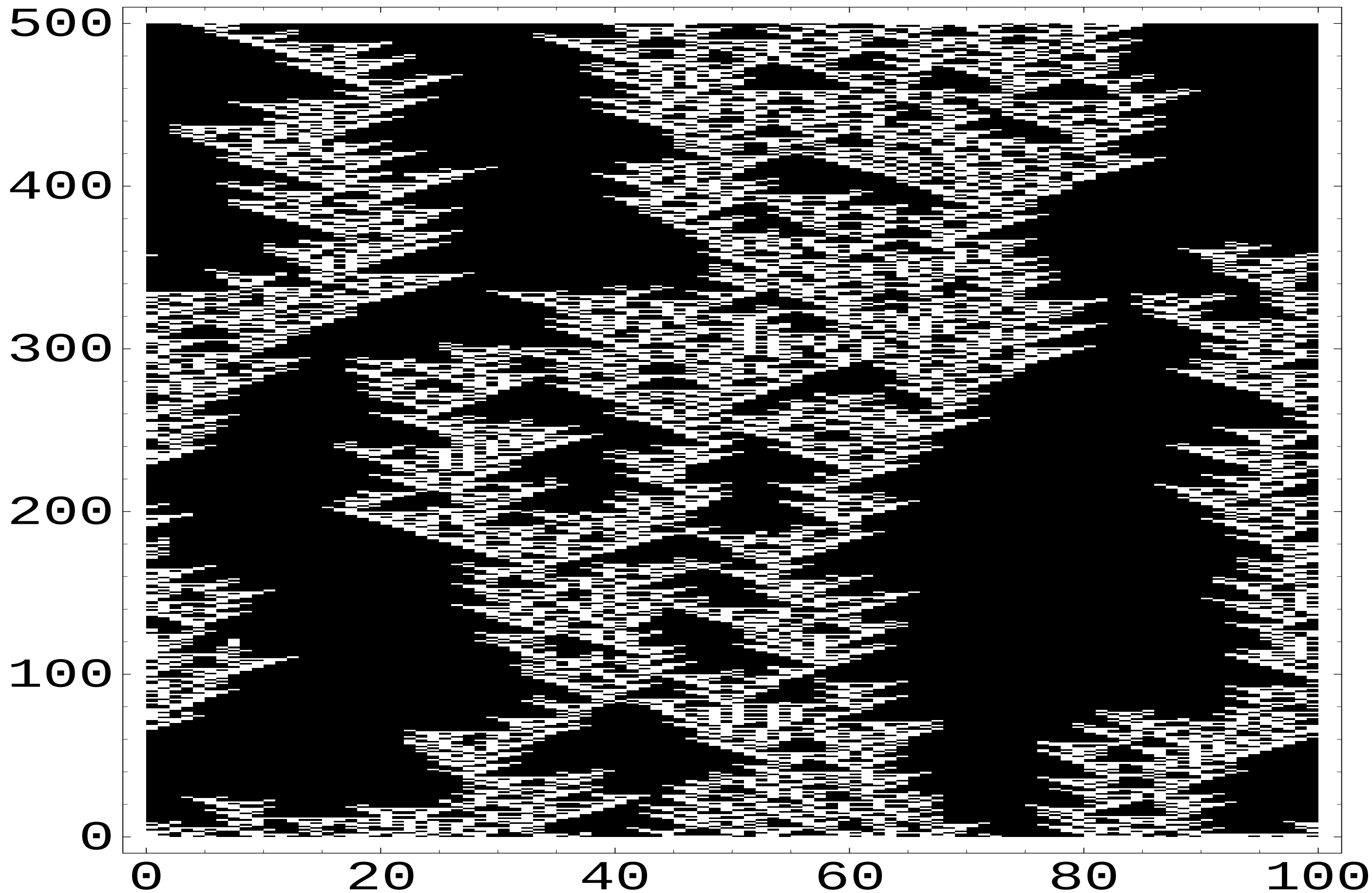
Finite probability of creation of bursts at a site even if all neighbours are laminar

Before onset of Type - II STI : there is spatial structure

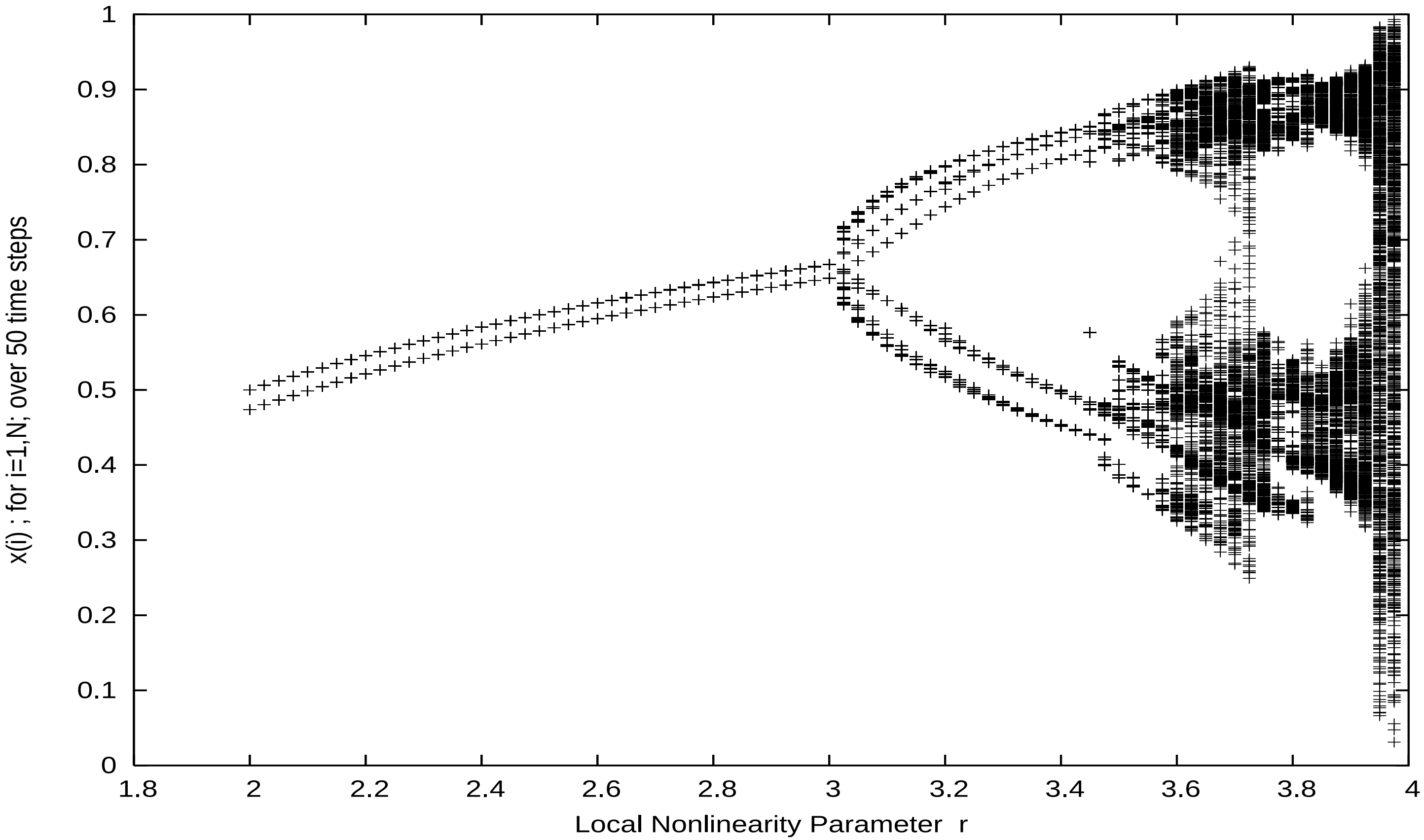
This type of STI has been observed as transition from local to global chaos


500
400
300
200
100
0
0
20
40
60
80
100

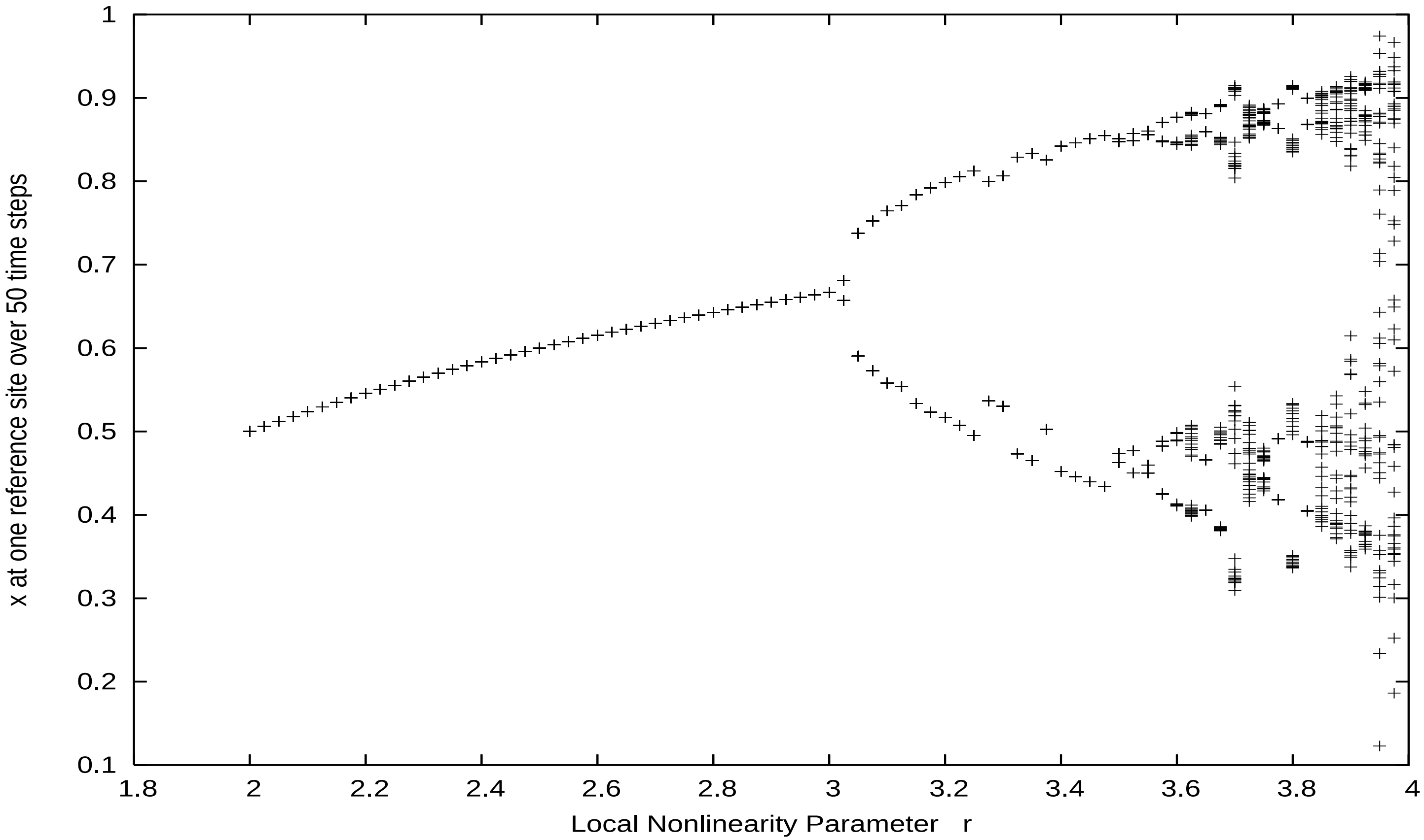
After Coarse-graining



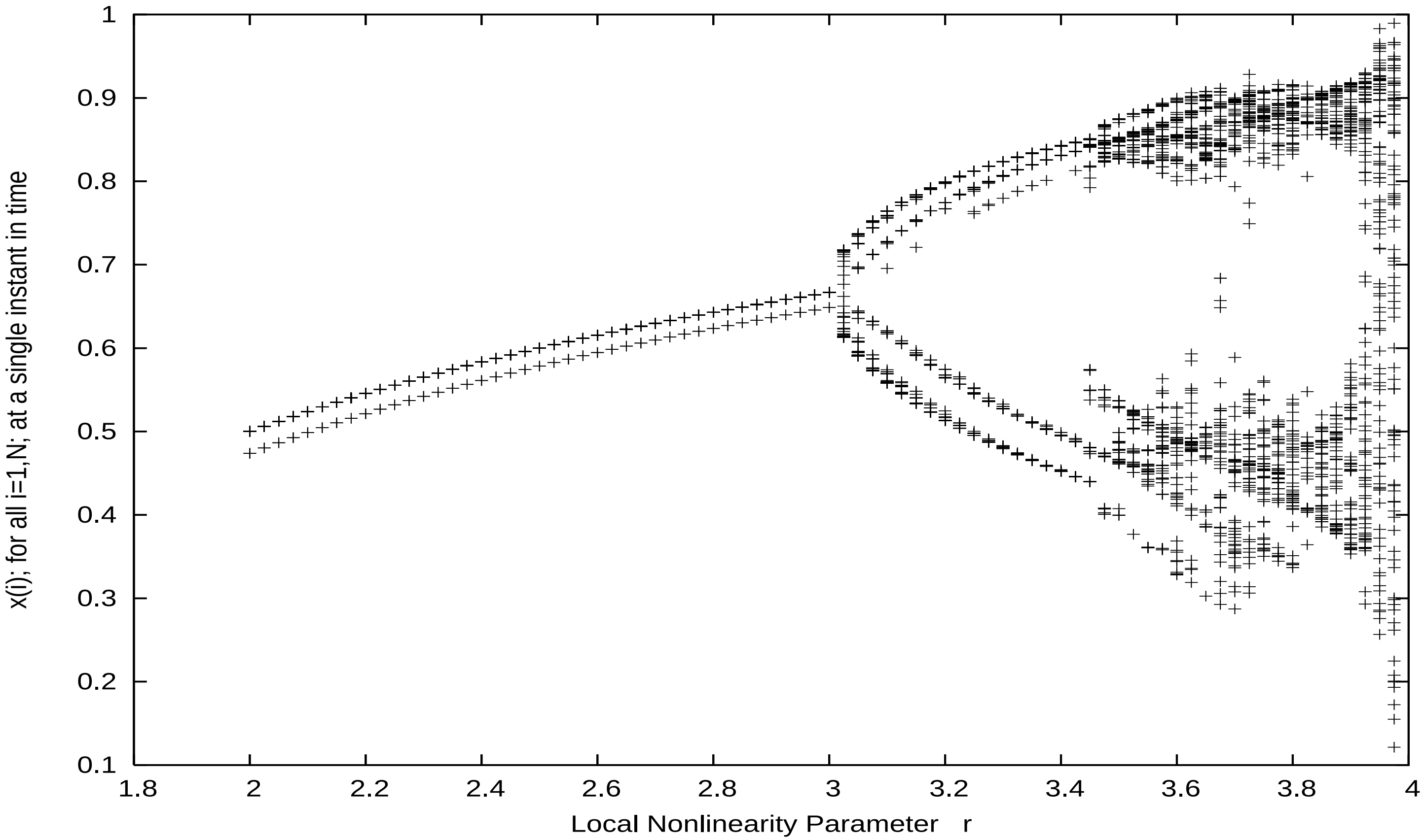
Bifurcation Diagram with respect to Local Nonlinearity



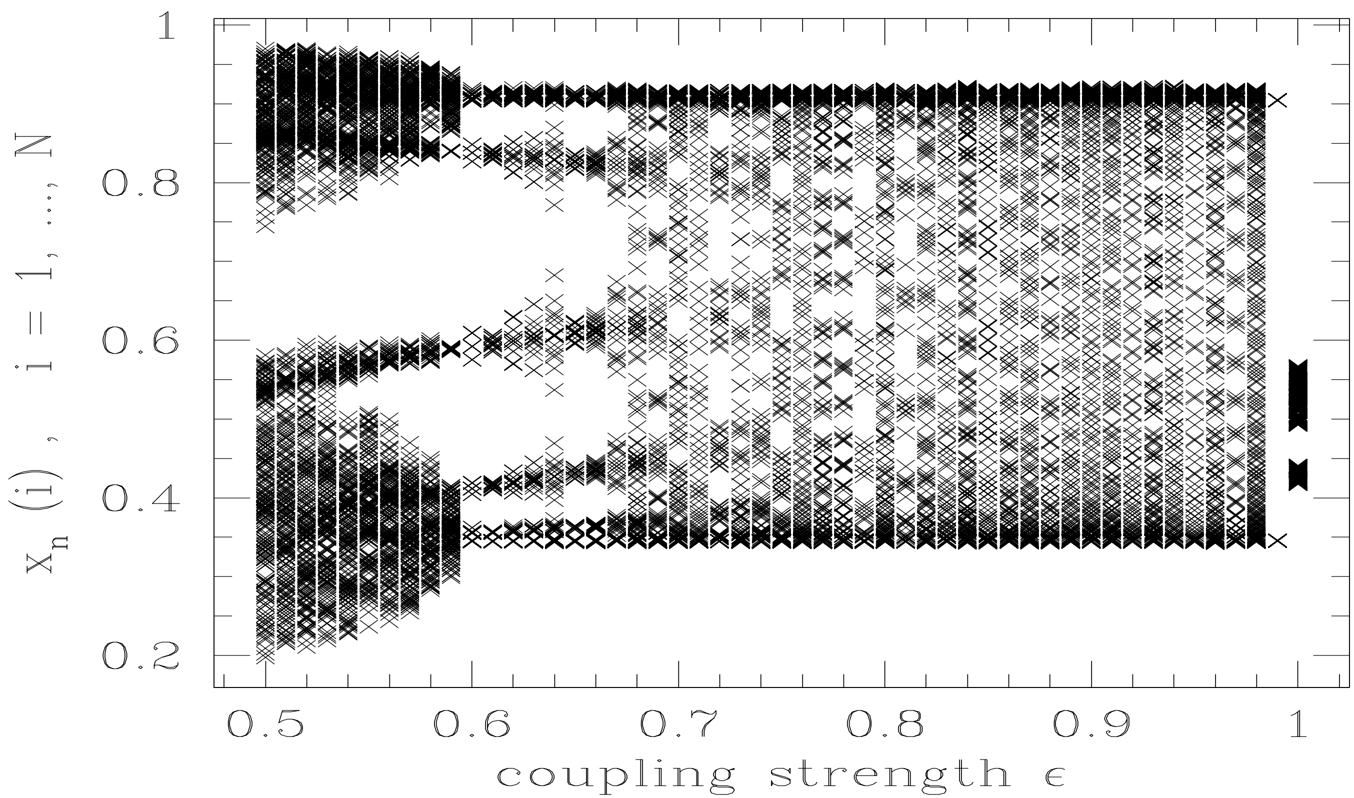
Bifurcation Diagram of the state at a single site with respect to Local Nonlinearity



Bifurcation Diagram of the state at a single snap-shot in time : with respect to Local Nonlinearity



Bifurcation Diagram with respect to coupling strength



Characterization

Visualization of dynamical phases

- Space-time plots
- Spatial return maps
- Time evolution of a generic site
- Bifurcation Diagram

Quantitative Characterization

- Fourier Transforms (Space and Time)
- Eigenvalue Spectrum
- N Lyapunov Exponents

$$N = \eta L^d$$

For a d-dimensional cubic lattice of side L with η local variables at each node

Given a set $\mathbf{y} = \mathbf{f}(\mathbf{x})$ of N equations in N variables $x_1, x_2 \dots x_N$ written explicitly as:

$$\mathbf{y} = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots f_N(\mathbf{x}))$$

or more explicitly as :

(1)
$$\begin{aligned} y_1 &= f_1(x_1, x_2, \dots x_N) \\ y_2 &= f_2(x_1, x_2, \dots x_N) \\ &. \\ &. \\ y_N &= f_N(x_1, x_2, \dots x_N) \end{aligned}$$

The Jacobian (matrix) is defined as:

$$J(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}.$$

Determinant of the Jacobian

$$J = \left| \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} \right|.$$

Linear Stability of the System

Determined by the Eigenvlaue spectrum of the Jacobian

Linear Stability Analysis of a spatiotemporal fixed point:

CML:

$$x^i_{n+1} = (1-\epsilon)f(x^i_n) + \frac{\epsilon}{2}(x^{i+1}_n + x^{i-1}_n)$$

Solution of spatiotemporal fixed point state x^*:

$$x^i_{n+1} = x^i_n = x^*$$

and

$x^j_n = x^i_n = x^*$ for all i, j

All $x_n(i) = x^*$ only when $x^* = f(x^*)$

For the case of the logistic map at $r = 4$:

Fixed point solution of the local map $x^* = 4x^*(1 - x^*) = 3/4$

Replacing $x_n(j) = x^* + h_n(j)$, and expanding to first order:

$$h_{n+1}(j) = (1-\epsilon)f'(x^*)h_n(j) + \frac{\epsilon}{2}\{h_n(j+1) + h_n(j-1)\}$$

For stability : Small perturbations $h(j)$ should decay with time

Vector $\mathbf{h}$ is the perturbation around the fixed point solution

$$\mathbf{h_{n+1}} = \{(1-\epsilon)f'(x^*)\,\mathbf{I} + \frac{\epsilon}{2}\,\mathbf{C}\}\,\mathbf{h_n}$$

where $\mathbf{I}$ is the identity (diagonal) matrix with 1 as diagonal entries

$\mathbf{C}$ is a circulant matrix with zero diagonal entries and 1's on the off-diagonal

Absolute value of the Eigenvalues of the above matrix should be bounded by 1 for stability

Confined in a disk of radius 1 in complex space

The first step is to find equilibria

The second step is to linearize the model at the equilibrium state, i.e., to estimate the Jacobian matrix

The third step is to estimate eigenvalues of this matrix

Condition of Stability:

Continuous models : Stable iff all eigenvalues have negative real parts

Discrete-time models are stable ifff all eigenvalues lie in the circle with the radius = 1 in the complex plane

Any algorithm to compute the lyapunov spectrum must contain 2 fundamental procedures:

1) Multiply by Jacobian at each step

2) Perform some kind of re-normalization

required to prevent the Jacobian matrix progressively getting more ill conditioned – until the largest lyapunov expoent swamps all the others

Gram-Schmidt orthogonalization

Exists N lyapunov exponents

Lyapunov spectrum is defined as the set of N lyapunov exponents arranged in decreasing order $\{\lambda_i\}_{i=1}^{N}$

The maximal lyapunov exponent determines the over-all behaviour

A lot of information in the distribution of the other lyapunov exponents as well

Kaplan-Yorke conjecture:

$$D_L = j + \frac{1}{|\lambda_{j+1}|} \sum_{i=1}^{j} \lambda_i$$

where j is the largest integer for which $\sum_{i=1}^{j} \lambda_i > 0$

Kolmogorov-Sinai (KS) entropy h bounded from above by the sum of the positive lyapunov exponents

$$h = \sum \lambda_i^+$$

KS entropy quantifies the mean rate of information production in a system, or alternately the mean rate of growth of uncertainty in a system subjected to small perturbations