Continuous Self Maps of Quadric Hypersurfaces

K. H. Paranjape and V. Srinivas

Introduction

Let G be a simply connected, semi-simple algebraic group over $\mathbf{C}$. Let $P \subset G$ be a parabolic subgroup, and let $Y = G/P$ be the homogeneous space. In a recent paper [KS], we showed that

> if $Y = G/P$ is as above and $f : Y \to Y$ is a finite (algebraic) self map of degree > 1, then $Y \cong \mathbf{P}^n$.

The paper arose out of an attempt to understand the following problem of Lazarsfeld [L]:

Problem -99 *Suppose G is a semi-simple algebraic group over $\mathbf{C}$, $P \subset G$ a maximal parabolic subgroup, $Y = G/P$. Let $f : Y \to X$ be a finite, surjective morphism of degree > 1 to a smooth variety X; then is $X \cong \mathbf{P}^n$? ($n = \dim X = \dim Y$)*

Lazarsfeld (*loc. cit.*) answers this in the affirmative when $Y = \mathbf{P}^n$, using the proof by S. Mori [M] of Hartshorne's conjecture. We also showed that Lazarsfeld's problem has an affirmative answer if Y is a smooth quadric hypersurface of dimension ≥ 3. This includes the case of the Grassmannian $Y = \mathbf{G}(2, 4)$. The general case seems to be open even for other Grassmann varieties.

Our goal in this paper is to study the analogous problems for continuous maps. Our homogeneous spaces are all complex submanifolds of complex projective spaces $\mathbf{CP}^n$, with the usual topology; we drop the $\mathbf{C}$ to simplify notation. We show

Theorem 1 *Let Q be a smooth quadric hypersurface in $\mathbf{P}^{n+1}$, where $n = 2k + 1$. Then for any positive integer $d \equiv 0 \pmod{2^k}$ there exist continuous maps $f : \mathbf{P}^n \to Q$ satisfying $f^*(\mathcal{O}_Q(1)) = \mathcal{O}_{\mathbf{P}^n}(d)$.*

Note that such maps have degree greater than one whenever $d > 1$. We also show

Theorem 2 *Let Q be a smooth quadric hypersurface in $\mathbf{P}^{n+1}$. Then there exists a positive integer m and continuous maps of degree $(m \cdot d)^n$ from Q to Q, for all $d \in \mathbf{N}$.*

Clearly Theorem 1 implies Theorem 2 in the case when n is odd, since there is an obvious map $Q \to \mathbf{P}^n$ of degree 2. However, we also construct self maps of Q of odd degree, when n is odd. Observe that the degree of any self map of Q is an n-th power.

Our proofs are by obstruction theory, using the standard (Bruhat) cell decomposition, and by induction on the dimension. The cases when n is even and odd are dealt with separately, and the induction is in steps of 2. From the computation of the homotopy groups of quadrics, such a division seems natural.

In the paper [KS], we had also proved:

Proposition 0 *Let $k \leq n$, $2 \leq l \leq m$ be integers, such that there exists a finite surjective morphism between Grassmann varieties*

$$f : \mathbf{G}(k, k+n) \to \mathbf{G}(l, l+m).$$

Then $k = l, m = n$ and f is an isomorphism.

We do not know whether there are continuous maps of degree bigger than one $f : \mathbf{G}(k, k+n) \to \mathbf{G}(l, l+m)$ with $k \neq l$, $m \neq n$ and $kn = lm$.

We wish to thank P. Polo and A. R. Shastri for stimulating discussions.

1 Preliminaries

1.1 Cell structure

We begin by recalling the cell decomposition of a quadric hypersurface in a form convenient for us. We will proceed by induction on dimension. The two smallest dimensions are :

$n = 1$: Q is $\mathbf{P}^1 \hookrightarrow \mathbf{P}^2$ as a conic and has a natural cell decomposition $Q = \mathbf{C} \cup \{\infty\}$. We have for $d \in \mathbf{N}$ maps $F_d : Q \to Q$ $(z \mapsto z^d)$ preserving the cell structure.

$n = 2$: Q is $\mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^2$ via the Segre embedding and has a natural cell decomposition

$$Q = (\mathbf{C} \times \mathbf{C}) \cup (\mathbf{C} \times \{\infty\}) \cup (\{\infty\} \times \mathbf{C}) \cup \{(\infty, \infty)\}.$$

For all $d \in \mathbf{N}$, we have maps $F_d : Q \to Q$ $((z, w) \mapsto (z^d, w^d))$ which preserve the cell structure.

We henceforth assume that $n \geq 3$.

Scholium 1 *Q has a cell decomposition with cells in each even (real) dimension (the Bruhat cell decomposition) such that we have:*

1. *For $n \geq 3$, $Q^{(2n-2)} = C$ is the projective cone over $Q' \subset \mathbf{P}^{n-1}$, a smooth quadric hypersurface of complex dimension $n-2$.*

2. *There is exactly one cell in each even (real) dimension except in dimension n for $n = 2k$.*

3. *$Q^{(n)}$ may be explicitly described as follows:*

   $n = 2k$: *$Q^{(n)} = L' \cup L''$, where $L', L'' \cong \mathbf{P}^k$ are linear subspaces of $\mathbf{P}^{n+1}$ and $L' \cap L'' = L \cong \mathbf{P}^{k-1}$.*

   $n = 2k+1$: *$Q^{(n)} = L \cong \mathbf{P}^k$ is a linear subspace of $\mathbf{P}^{n+1}$.*

4. *For $n \geq 3$, $Q^{(n+2)} \subset C$ is the Thom space over $Q'^{(n)}$ of the complex line bundle $\mathcal{O}_{Q'^{(n)}}(1)$.*

Proof: Let p be any point of Q and let $H \subset \mathbf{P}^{n+1}$ be a hyperplane tangent to Q at p. Then $Q \cap H = C$ is the projective cone over $Q' \subset \mathbf{P}^{n-1}$, a smooth quadric hypersurface of complex dimension $n-2$ (i.e. C is the Thom space of the complex line bundle $\mathcal{O}_{Q'}(1) = \mathcal{O}_{\mathbf{P}^{n-1}}(1) \mid_{Q'}$). By induction on dimension we get a cell decomposition of Q'. If $Q'^{(m)}$ is the m-skeleton of Q', and $C(Q'^{(m)})$ is the Thom space of $\mathcal{O}_{Q'^{(m)}}(1)$, then $C(Q'^{(m)}) - C(Q'^{(m-1)})$ is a union of cells of dimension $m+2$. Thus, we obtain a cell decomposition of C. Since $Q - C = \mathbf{C}^n$, we obtain the desired cell structure. □

We shall use the following construction.

Construction 1 Let X, Y be compact topological spaces, L and M complex line bundles on X and Y respectively. Let $f : X \to Y$ be a continuous map, such that there is an isomorphism $\varphi : L^{\otimes d} \to f^*(M)$, for some positive integer d. Then there exists a map $\Phi : L \to M$ giving a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\Phi} & M \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where Φ is the d-th power map on fibres of the vertical arrows. The restriction of Φ to the S^1-bundles $\widetilde{X}$, $\widetilde{Y}$ of L, M respectively induces a map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$, which has degree d along the fibres. If $C(X, L)$ and

$C(Y, M)$ denote the Thom spaces of L and M respectively, we have a map $C(f) = C(f, \varphi) : C(X, L) \to C(Y, M)$ induced by Φ.

In particular, if $X \subset \mathbf{P}^N$ is a projective variety, $f : X \to X$ an algebraic self-map, $L = M = \mathcal{O}_X(1)$, and φ is an isomorphism of algebraic line bundles, then $C(X, L)$ is the projective cone (in $\mathbf{P}^{N+1}$) of X and $C(f)$ can be regarded as an algebraic self-map of $C(X, L)$. Note that φ is unique upto a scalar multiple.

We had noted the existence of morphisms $F_d : Q \to Q$ for each $d > 0$, for a smooth quadric Q of dimension one or two. These maps satisfy $F_{d_1} \circ F_{d_2} = F_{d_1 d_2}$ for all $d_1, d_2 > 0$. By repeatedly applying the above constructions, we obtain

Lemma 2 *For each $d > 0$, there is an algebraic morphism*

$$F_{d,n} : Q^{(n+2)} \to Q^{(n+2)}$$

and an algebraic isomorphism

$$\varphi_{d,n} : \mathcal{O}_{Q^{(n+2)}}(d) \to F^*_{d,n}(\mathcal{O}_{Q^{(n+2)}}(1))$$

such that

(i) under the identification (by Scholium 1)

$$Q^{(n+2)} = C(Q'^{(n)}, \mathcal{O}_{Q'^{(n)}}(1))$$

we have

$$F_{d,n} = C(F_{d,n-2}, \varphi_{d,n-2})$$

.

(ii) $F_{d_1,n} \circ F_{d_2,n} = F_{d_1 d_2,n}$.

□

If $F_{d,n-2} : Q'^{(n)} \to Q'^{(n)}$ extends to a continuous map $f' : Q' \to Q'$, then the isomorphism $\varphi_{d,n-2}$ can be extended to an isomorphism (of topological complex line bundles)

$$\varphi' : \mathcal{O}_{Q'}(d) \to f'^*(\mathcal{O}_{Q'}(1)).$$

Then the induced continuous map $C(f') = C(f', \varphi') : C \to C$ restricts to $F_{d,n}$ on $Q^{(n+2)} \subset C$. Since Q is obtained from C by attaching $\mathbf{C}^n$ via a map $a : S^{2n-1} \to C$, the map $C(f') : C \to C$ extends to a map $f : Q \to Q$ if and only if $C(f')_*([a]) = m[a] \in \pi_{2n-1}(C)$, for some $m \in \mathbf{Z}$. Thus, we need to compute $\pi_{2n-1}(C)$, and the action of $C(f')_*$ on it.

1.2 Computation of Homotopy groups

We have the S^1-fibration (Hopf fibration) $S^{2n+3} \to \mathbf{P}^{n+1}$. Let $\widetilde{Q} \to Q$ be the restriction of this to Q.

Scholium 3 *$\widetilde{Q}$ is the total space of the unit sphere bundle of the tangent bundle of S^{n+1}. Hence we have a fibration*

$$S^n \to \widetilde{Q} \to S^{n+1}. \qquad (*)$$

Proof: If $(z_0 : \cdots : z_n)$ are homogeneous coordinates on $\mathbf{P}^{n+1}$, $u_i = \Re(z_i)$ and $v_i = \Im(z_i)$, then we may take

$$S^{2n+3} = \{(u_i, v_i) \mid \sum u_i^2 + \sum v_i^2 = 2\}.$$

We may assume that Q is defined in these coordinates by the equation $\sum z_i^2 = 0$, so that

$$\widetilde{Q} = \{(u_i, v_i) \mid \sum u_i^2 = \sum v_i^2 = 1 \text{ and } \sum u_i v_i = 0\}.$$

Hence, the projection to the u_i's is a fibration of the required sort. $\square$

For any subset $A \subset Q$, let $\widetilde{A}$ denote its inverse image in $\widetilde{Q}$, so that there is an induced S^1-fibration $\widetilde{A} \to A$. For any abelian group G let ${}_2G$ denote its 2-torsion subgroup and let $G/2 = G \otimes \mathbf{Z}/2\mathbf{Z}$.

Lemma 4 *(i) If n is* odd, *then for $i \leq 2n-1$, we have a short exact sequence*

$$0 \to \pi_i(S^n)/2 \to \pi_i(\widetilde{Q}) \to {}_2\pi_i(S^{n+1}) \to 0.$$

(ii) If n is even, *then we have a split exact sequence for each i*

$$0 \to \pi_i(S^n) \to \pi_i(\widetilde{Q}) \to \pi_i(S^{n+1}) \to 0.$$

Proof: In the long exact sequence

$$\cdots \to \pi_i(S^n) \to \pi_i(\widetilde{Q}) \to \pi_i(S^{n+1}) \xrightarrow{\partial_i} \pi_{i-1}(S^n) \to \cdots$$

the boundary maps ∂_i are the maps on π_i induced by a map (well defined upto homotopy) $\Delta : \Omega S^{n+1} \to S^n$ coming from the fibration $(*)$. Suppose $s : S^n \to \Omega S^{n+1} = \Omega\Sigma S^n$ is the map inducing the suspension homomorphisms $\Sigma_i : \pi_i(S^n) \to \pi_{i+1}(S^{n+1})$. Since $(*)$ is the spherical fibration associated to the tangent bundle of S^{n+1}, it is well known (see [W] IV, (10.4)) that if $\theta = \Delta \circ s$,

$$[\theta] = 1 + (-1)^{n+1} \in \pi_n(S^n)$$

where $1 \in \pi_n(S^n)$ is the standard generator. From the Freudenthal Suspension theorem, Σ_i is an isomorphism for $i \le 2n-2$, and Σ_{2n-1} is a surjection.

We shall often make use of the following well known result (see [W] XI, (1.11), (1.12), (1.16)).

Scholium 5 *Let $f : S^n \to S^n$ be any continuous map of degree d, where $n > 1$. Then the induced map*

$$f_* : \pi_i(S^n) \to \pi_i(S^n)$$

is

(i) multiplication by d if $i < 2n-1$

(ii) multiplication by d on the torsion subgroup, for $i = 2n-1$ (and in particular on π_{2n-1}, if n is odd)

(iii) multiplication by d^2 on $\pi_{2n-1}(S^n) \otimes \mathbf{Q}$.

□

If n is *even*, then $[\theta] = 0$ so that $\theta_* = 0$. Hence, ∂_{n+1} vanishes. Thus $(*)$ has a homotopy section and the long exact sequence for this fibration splits into short exact sequences as asserted in *(ii)*.

If n is *odd*, Scholium 5 implies that θ_* acts as multiplication by 2 on $\pi_i(S^n)$ in the range $i \le 2n-1$. This proves *(i)*. □

2 The odd dimensional case

We begin by recalling the statement of Theorem 1.

Theorem 1 *Let Q be a smooth quadric hypersurface in $\mathbf{P}^{n+1}$, where $n = 2k+1$. Then for any positive integer $d \equiv 0 \pmod{2^k}$ there exist continuous maps $f : \mathbf{P}^n \to Q$, where $f^*(\mathcal{O}_Q(1)) = \mathcal{O}_{\mathbf{P}^n}(d)$.*

Proof: This is obvious for $n = 1$. We may assume, by induction, that the theorem holds for quadrics of dimension $n-2$.

$\mathbf{P}^n$ has a cell decomposition where the skeleta are linear projective subspaces of smaller dimension. By Scholium 1 we have a cell decomposition of Q whose $n-1$ skeleton is the projective cone C over a smooth quadric Q' of dimension $n-2$. For any $d' \equiv 0 \pmod{2^{k-1}}$, the induction hypothesis gives us a map $f' : \mathbf{P}^{n-2} \to Q'$ satisfying $f'^*(\mathcal{O}_{Q'}(1)) = \mathcal{O}_{\mathbf{P}^{n-2}}(d')$. Since $\mathbf{P}^{n-1}$ is the projective cone over $\mathbf{P}^{n-2}$, Construction 1.1 yields a map $C(f') : \mathbf{P}^{n-1} \to C$.

The obstruction to extending this map to a map $\mathbf{P}^n \to Q$ is a class $O(C(f')) \in H^{2n}(\mathbf{P}^n, \pi_{2n-1}(Q))$. By Lemma 4, the group $\pi_{2n-1}(Q)$ has exponent 4. Let $\widetilde{f}$ be the composite

$$\mathbf{P}^{n-1} \xrightarrow{\alpha} \mathbf{P}^{n-1} \xrightarrow{C(f')} C,$$

where α is the restriction of the map $\beta : \mathbf{P}^n \to \mathbf{P}^n$ preserving the cell structure, and given in suitable homogeneous coordinates by

$$(z_0 : \cdots : z_n) \mapsto (z_0^2 : \cdots : z_n^2).$$

The obstruction to extending $\widetilde{f}$ to a map $\mathbf{P}^n \to Q$ is

$$O(\widetilde{f}) = \beta^*(O(C(f'))) \in H^{2n}(\mathbf{P}^n, \pi_{2n-1}(Q)).$$

But clearly $\beta^* = 0$ on this cohomology group.

Let $f : \mathbf{P}^n \to Q$ be an extension of $\widetilde{f}$. We have a commutative diagram of integral cohomology groups

$$\begin{array}{ccc} H^2(Q) & \xrightarrow{f^*} & H^2(\mathbf{P}^n) \\ \downarrow \wr & & \downarrow \wr \\ H^2(Q') & \xrightarrow{f''^*} & H^2(\mathbf{P}^{n-2}) \end{array}$$

where f'' is the composite of f' with the restriction of β. Since f''^* is multiplication by $2d'$, so is f^*. $\square$

We now prove the following refinement of Theorem 2, in the odd dimensional case.

Theorem $2'$ *Let $Q \subset \mathbf{P}^{n+1}$ be a smooth quadric hypersurface with $n = 2k+1$. Then there exists a continuous map $f : Q \to Q$ of degree d^n whenever*

- *(i)* $d \equiv 0 \pmod{2^k}$, *or*
- *(ii)* $d = e^{2^{n-1}}$, *for some integer e.*

Proof: Clearly (i) follows from Theorem 1. We prove (ii) by showing that for the chosen integers d the map $F_{d,n} : Q^{(n+2)} \to Q^{(n+2)}$ extends to a map $f : Q \to Q$. By induction, we may assume that $F_{d,n-2} : Q'^{(n)} \to Q'^{(n)}$ extends to a map $f' : Q' \to Q'$, for Q' a smooth quadric hypersurface of dimension $n-2$, and $d = e^{2^{n-3}}$ for some $e > 0$. We have constructed (see (1.1)) a map $C(f') : C \to C$ which satisfies $C(f') \mid_{Q^{(n+2)}} = F_{d,n}$. By computing $C(f')_*$ on $\pi_{2n-1}(C)$ we will show that the obstruction to extending the four-fold composite $C(f')^4$ to a self map of Q vanishes.

There is a filtration F on $\pi_{2n-1}(C)$ given as follows. Take $F_0 = \pi_{2n-1}(C)$; F_1 is the kernel of the composite

$$\pi_{2n-1}(C) \xrightarrow{\cong} \pi_{2n-1}(\widetilde{C}) \xrightarrow{\gamma} H_{2n-1}(\widetilde{C}, \mathbf{Z}),$$

where γ is the Hurewicz map; and finally $F_2 = \mathrm{im}\,(\pi_{2n-1}(Q^{(n)}) \to \pi_{2n-1}(C))$.

Clearly $C(f')_*$ is compatible with this filtration and induces a map $\mathrm{gr}^F C(f')_*$ on $\mathrm{gr}^F \pi_{2n-1}(C)$. This map may be computed as follows.

Lemma 6 *(i) $F_2, F_1/F_2$ are vector spaces over $\mathbf{Z}/2$.*

(ii) The natural composite map

$$\pi_{2n}(Q, C) \to \pi_{2n-1}(C) \to F_0/F_1 \hookrightarrow H_{2n-1}(\widetilde{C}, \mathbf{Z})$$

is an isomorphism, giving a direct sum decomposition

$$\pi_{2n-1}(C) \cong F_1 \oplus \pi_{2n}(Q, C).$$

(iii) $\mathrm{gr}^F C(f')_*$ is multiplication by d^n.

Proof: We have a commutative diagram with exact bottom row

$$\begin{array}{ccccccc}
 & H_{2n}(\widetilde{Q}, \widetilde{C}) & \xrightarrow{\partial} & H_{2n-1}(\widetilde{C}, \mathbf{Z}) & & & \\
 & \alpha \uparrow & & \uparrow \gamma & & & \\
0 \to & \pi_{2n}(\widetilde{Q}, \widetilde{C}) & \to & \pi_{2n-1}(\widetilde{C}) & \to \pi_{2n-1}(\widetilde{Q}) \to 0 & &
\end{array}$$

The boundary homomorphism ∂ is an isomorphism by the long exact sequence of homology for the pair $(\widetilde{Q}, \widetilde{C})$, and α is an isomorphism by the relative Hurewicz theorem. This proves (ii) and gives an isomorphism $F_1 \cong \pi_{2n-1}(\widetilde{Q})$. The self map $\widetilde{C(f')}$ of $\widetilde{C}$ is of degree d^n and this gives (iii) for F_0/F_1.

Since $Q^{(n)} = L$ is a linear projective subspace of $\mathbf{P}^{n+1}$, $\widetilde{L}$ is an S^n in $\widetilde{Q}$; it is easy to check that, in the fibration $(*)$, this maps isomorphically to a great sphere $S \subset S^{n+1}$. In fact $\widetilde{L}$ is a section of the sphere bundle of the tangent bundle of S which is contained in $\widetilde{Q}$ by Scholium 3. Let D^- be a hemisphere capping S in S^{n+1} and let U be its inverse image in $\widetilde{Q}$. We have the following

Sublemma 7 *Let $i : S^n \to \widetilde{Q}$ be a fibre of $(*)$ lying over a point of D^-. A unit tangent vector field v on $S \cong S^n$ gives a map $v : S^n \to \widetilde{Q}$ which is homotopic within U to the inclusion i.*

Proof: Let p be the point of D^- orthogonal to S (i.e. the "pole"). We have a map $\ell : S \times [0, \pi] \to D^-$ given by

$$(x, t) \mapsto \sin(t) \cdot p + \cos(t) \cdot x.$$

For all $(x, t) \in S \times [0, \pi]$ let $n(x, t)$ be the tangent vector at the point $\ell(x, t)$ given by $\sin(t) \cdot x - \cos(t) \cdot p$. Then, $d\ell_{(x,t)}v(x)$ is orthogonal to $n(x, t)$ in the tangent space of S^{n+1} at $\ell(x, t)$ so that we get a map $H : S \times [0, \pi] \to \widetilde{Q}$ given by the formula

$$(x, t) \mapsto d\ell_{(x,t)}v(x) + \sin(t) \cdot n(x, t).$$

Clearly $H(x, 0) = v(x)$ and $H(x, t) = x$ considered as a tangent vector at p. $\square$

Thus we have isomorphisms

$$F_2 \cong \mathrm{im}\,(\pi_{2n-1}(\widetilde{L}) \to \pi_{2n-1}(\widetilde{Q})) = \mathrm{im}\,(\pi_{2n-1}(S^n) \to \pi_{2n-1}(\widetilde{Q})),$$

where $S^n \to Q$ is the inclusion of the fibre of $(*)$. Hence F_2 is a vector space over $\mathbf{Z}/2$, and by Scholium 5 the action of $\mathrm{gr}^F C(f')_*$ on it is by $d^{k+1} \equiv d^n \pmod 2$. Further, we obtain an isomorphism

$$F_1/F_2 \cong \pi_{2n-1}(\widetilde{Q})/\mathrm{im}\,(\pi_{2n-1}(\widetilde{L})) \cong {}_2(\pi_{2n-1}(S^{n+1}, D^-)),$$

so that F_1/F_2 is a $\mathbf{Z}/2$-vector space.

Let $g : (D^{n+1}, S^n) \to (\widetilde{Q^{(n+1)}}, \widetilde{L})$ be the generator of $\pi_{n+1}(\widetilde{Q^{(n+1)}}, \widetilde{L}) \cong \pi_{n+1}(Q^{(n+1)}, L) \cong \mathbf{Z}$. We have a diagram, commutative upto homotopy,

$$\begin{array}{ccc} (D^{n+1}, S^n) & \to & (\widetilde{Q^{(n+1)}}, \widetilde{L}) \\ \varphi_d \downarrow & & \downarrow \widetilde{F_{d,n}} \\ (D^{n+1}, S^n) & \to & (\widetilde{Q^{(n+1)}}, \widetilde{L}) \end{array}$$

where φ_d is a map of degree d^{k+1}. From the Scholium 5 we see that $(\varphi_d)_*$ induces multiplication by d^{k+1} on $\pi_{2n-1}(D^{n+1}, S^n)$.

By the sublemma we have isomorphisms

$$\pi_{n+1}(\widetilde{Q^{(n+1)}}, \widetilde{L}) \stackrel{\cong}{\to} \pi_{n+1}(\widetilde{Q}, \widetilde{L}) \stackrel{\cong}{\to} \pi_{n+1}(S^{n+1}, D^-),$$

so that the composite $\rho : (D^{n+1}, S^n) \to (S^{n+1}, D^-)$ of g and the natural map $(\widetilde{Q^{(n+1)}}, \widetilde{L}) \to (S^{n+1}, D^-)$ is also a generator for $\pi_{n+1}(S^{n+1}, D^-)$. By the Freudenthal suspension theorem, ρ_* is an isomorphism on π_{2n-1}. From the diagram

$$\begin{array}{ccccc} & H_{2n}(\widetilde{Q}, \widetilde{C}; \mathbf{Z}) & \stackrel{\partial}{\to} & H_{2n-1}(\widetilde{C}, \widetilde{L}; \mathbf{Z}) & \\ & \alpha \uparrow & & \uparrow \gamma & \\ 0 \to & \pi_{2n}(\widetilde{Q}, \widetilde{C}) & \to & \pi_{2n-1}(\widetilde{C}, \widetilde{L}) & \to \pi_{2n-1}(\widetilde{Q}, \widetilde{L}) \to 0 \end{array}$$

where α, ∂ are isomorphisms,we see that

$$\mathrm{im}\,(\pi_{2n-1}(D^{n+1}, S^n) \to \pi_{2n-1}(\widetilde{C}, \widetilde{L})) = \ker(\pi_{2n-1}(\widetilde{C}, \widetilde{L}) \to H_{2n-1}(\widetilde{C}, \widetilde{L}; \mathbf{Z})).$$

In particular, F_1/F_2 is contained in this image; thus $C(f')_*$ acts by multiplication by $d^{k+1} \equiv d^n \pmod 2$ on F_1/F_2,and this completes the proof of (iii). $\square$

From this lemma we see that we have $\varphi_d \in \mathrm{Hom}\,(\pi_{2n}(Q, C), F_1) = F_1$ and $\psi_d \in \mathrm{Hom}\,(F_1/F_2, F_2) \subset \mathrm{End}\,(F_1)$ such that, for all pairs (a, b) in $\pi_{2n-1}(C) = \pi_{2n}(Q, C) \oplus F_1$ we have the equation

$$C(f')_*(a, b) = (d^n a, d^n b + \psi_d(b) + \varphi_d(a)).$$

Since both $F_1, F_1/F_2$ are of exponent 2, it follows that the four-fold composite of $C(f')$ satisfies

$$C(f')^4_*(a, 0) = (d^{4n} a, 0),$$

and this proves the theorem. $\square$

3 The even dimensional case

In this section we prove the following theorem

Theorem 2 *Let Q be a smooth quadric hypersurface of dimension $n = 2k$. There is an integer m such that for any positive $d \equiv 0 \pmod{m}$, the map $F_{d,n} : Q^{(n+2)} \to Q^{(n+2)}$ extends to a continuous map $f : Q \to Q$ (of degree d^n).*

Proof: Suppose that for some $d > 0$, the map $F_{d,n-2} : Q'^{(n)} \to Q'^{(n)}$ extends to a map $f' : Q' \to Q'$. Then the map $C(f') : C \to C$, obtained by the construction of (1.1), restricts to $F_{d,n} : Q^{(n+2)} \to Q^{(n+2)}$. We compute below the obstruction to extending $C(f')$ to a map $f : Q \to Q$.

As in the odd dimensional case, we begin by observing that we have the diagram with exact bottom row

$$\begin{array}{ccccc} & H_{2n}(\widetilde{Q}, \widetilde{C}) & \stackrel{\partial}{\to} & H_{2n-1}(\widetilde{C}, \mathbf{Z}) & \\ & \alpha \uparrow & & \uparrow \gamma & \\ 0 \to & \pi_{2n}(\widetilde{Q}, \widetilde{C}) & \to & \pi_{2n-1}(\widetilde{C}) & \to \pi_{2n-1}(\widetilde{Q}) \to 0 \end{array}$$

where the vertical maps are Hurewicz maps. Since ∂, α are isomorphisms, the composite

$$\pi_{2n}(Q, C) \to \pi_{2n-1}(C) \cong \pi_{2n-1}(\widetilde{C}) \xrightarrow{\gamma} H_{2n-1}(\widetilde{C})$$

is an isomorphism, and

$$\pi_{2n-1}(C) \cong \pi_{2n}(Q, C) \oplus \pi_{2n-1}(Q) \cong \pi_{2n}(\widetilde{Q}, \widetilde{C}) \oplus \pi_{2n-1}(\widetilde{Q}).$$

From Lemma 4, we have a split exact sequence for each i

$$0 \to \pi_i(S^n) \to \pi_i(\widetilde{Q}) \to \pi_i(S^{n+1}) \to 0,$$

where the splitting is obtained from a homotopy section of $(*)$. Since $Q^{(n)} = Q^{(n+1)}$, the map $\pi_i(Q^{(n)}) \to \pi_i(Q)$ is an isomorphism for $i = n$, and a surjection for $i = n + 1$. In particular, the homotopy section of $(*)$, and the inclusion of the fibre $S^n \subset \widetilde{Q}$ of $(*)$, factor through $Q^{(n)}$. Hence $\pi_{2n-1}(Q^{(n)}) \to \pi_{2n-1}(Q)$ is surjective. Thus

$$\pi_{2n-1}(C) \cong \pi_{2n}(Q, C) \oplus \mathrm{im}\,(\pi_{2n-1}(Q^{(n)})).$$

We can refine this a little. Since $\pi_n(Q^{(n)}) \cong \pi_n(Q) \cong \mathbf{Z}$, there is a map $\widetilde{g} : S^n \to \widetilde{Q^{(n)}}$, inducing a map $g : S^n \to Q^{(n)}$, such that $\widetilde{g}$, g represent generators of π_n, and $\widetilde{g}$ is homotopic to the inclusion of the fibre of $(*)$; further,

$$\pi_n(\widetilde{Q^{(n)}}) \cong H_n(\widetilde{Q^{(n)}}, \mathbf{Z}) \hookrightarrow H_n(Q^{(n)}, \mathbf{Z}),$$

and $(F_{d,n})_*$ acts by multiplication by d^k on $H_n(Q^{(n)}, \mathbf{Z})$. Then there is an inclusion $h : \pi_{2n-1}(S^n) \hookrightarrow \pi_{2n-1}(C)$ induced by g and a homotopy commutative diagram

$$\begin{array}{ccc} S^n & \stackrel{\mu}{\to} & S^n \\ g \downarrow & & \downarrow g \\ Q^{(n)} & \stackrel{F_{d,n}}{\to} & Q^{(n)} \end{array}$$

where μ has degree d^k.

Next, an easy computation shows that $\widetilde{L'} \cong S^{n+1} \subset \widetilde{Q^{(n)}} \subset \widetilde{Q}$ maps isomorphically onto S^{n+1} in the fibration $(*)$. Further $F_{d,n}$ restricts to a self map of L' of degree d^k.

Thus we have a decomposition

$$\mathrm{im}\,(\pi_{2n-1}(Q^{(n)}) \to \pi_{2n-1}(C)) = h_*(\pi_{2n-1}(S^n)) \oplus \pi_{2n-1}(L').$$

The action of $C(f')_*$ on the left is compatible with this decomposition, and induces μ_* on $\pi_{2n-1}(S^n)$, and $(F_{d,n})_*$ on $\pi_{2n-1}(L')$. Note that since n is even, $\pi_{2n-1}(S^n) \cong \mathbf{Z} \oplus \pi_{2n-1}(S^n)_{tors}$ (where the subscript "tors" denotes the torsion subgroup).

Lemma 8 *(i)* *$C(f')_*$ acts by multiplication by d^n on $\pi_{2n-1}(C) \otimes \mathbf{Q}$.*

(ii) *With respect to the direct sum decomposition*

$$\pi_{2n-1}(C) = \pi_{2n}(Q, C) \oplus (\mathbf{Z} \oplus \pi_{2n-1}(S^n)_{tors}) \oplus \pi_{2n-1}(L'),$$

$C(f')_*$ has a matrix of the form

$$\begin{pmatrix} d^n & 0 & 0 & 0 \\ 0 & d^n & 0 & 0 \\ \varphi_1 & \varphi_3 & d^k & 0 \\ \varphi_2 & 0 & 0 & d^{k+1} \end{pmatrix}$$

where

$$\begin{array}{lcl} \varphi_1 & \in & \mathrm{Hom}\,(\pi_{2n}(Q, C), \pi_{2n-1}(S^n)_{tors}) \\ \varphi_2 & \in & \mathrm{Hom}\,(\pi_{2n}(Q, C), \pi_{2n-1}(S^{n+1})) \\ \varphi_3 & \in & \mathrm{Hom}\,(\mathbf{Z}, \pi_{2n-1}(S^n)_{tors}) \end{array}$$

Proof: From the Scholium 5, the action of μ_* on $\pi_{2n-1}(S^n)_{tors}$ is by d^k, while it is by $d^{2k} = d^n$ on $\pi_{2n-1}(S^n) \otimes \mathbf{Q}$. The action of $(F_{d,n})_*$ on

$$\pi_{n+1}(\widetilde{L'}) \cong H_{n+1}(\widetilde{L'}, \mathbf{Z}) \cong H_n(L', H_1(S^1))$$

is by d^{k+1}. From the Scholium, this implies that $(F_{d,n})_*$ acts by d^{k+1} on $\pi_{2n-1}(L')$. Hence (ii) follows, once we prove (i).

Now $Q^{(n)} = L' \cup L''$ where $L' \cap L'' = L \cong \mathbf{P}^{k-1}$. Since $\pi_n(L)$ is finite, we see that $\pi_n(Q^{(n)}) \to \pi_n(Q^{(n)}, L)$ is injective. We have a diagram, whose vertical arrows are Hurewicz maps,

$$\begin{array}{ccc} H_n(Q^{(n)}, \mathbf{Z}) & \stackrel{\cong}{\to} & H_n(Q^{(n)}, L; \mathbf{Z}) \\ \uparrow & & \uparrow \wr \\ \pi_n(Q^{(n)}) & \to & \pi_n(Q^{(n)}, L) \end{array}$$

so that the Hurewicz map on $\pi_n(Q^{(n)})$ is injective.

Consider the quotient map

$$Q^{(n)} = L' \cup L'' \to (L' \cup L'')/L \cong S^n \vee S^n.$$

This induces an isomorphism on H_n, and hence an injection on π_n. We have a diagram

$$\begin{array}{ccc} Q^{(n)} & \xrightarrow{F_{d,n}} & Q^{(n)} \\ \downarrow & & \downarrow \\ S^n \vee S^n & \xrightarrow{\rho} & S^n \vee S^n \end{array}$$

where $\rho = \rho' \vee \rho''$, and ρ', ρ'' are self maps of S^n of degree d^k.

Let $(S^n \vee S^n) \otimes \mathbf{Q}$ be the space obtained from $S^n \vee S^n$ by localising at $\mathbf{Q}$. The map $Q^{(n)} \to (S^n \vee S^n) \otimes \mathbf{Q}$ extends to a map $\psi : C \to (S^n \vee S^n) \otimes \mathbf{Q}$, since $\pi_i(S^n \vee S^n) \otimes \mathbf{Q} = 0$ for $n < i < 2n-1$. Further, the diagram

$$\begin{array}{ccc} C & \stackrel{C(f')}{\to} & C \\ \psi \downarrow & & \downarrow \psi \\ S^n \vee S^n \otimes \mathbf{Q} & \stackrel{\rho}{\to} & S^n \vee S^n \otimes \mathbf{Q} \end{array}$$

commutes upto homotopy, since there are no obstructions to extending the constant homotopy on $Q^{(n)}$.

The map $\psi_* : \pi_{2n-1}(C) \otimes \mathbf{Q} \to \pi_{2n-1}((S^n \vee S^n) \otimes \mathbf{Q})$ is injective on the summand

$$(\pi_{2n-1}(S^n) \oplus \pi_{2n-1}(L')) \otimes \mathbf{Q} = \pi_{2n-1}(S^n) \otimes \mathbf{Q}$$

by construction. Hence the map

$$\nu : \pi_{2n-1}(C) \otimes \mathbf{Q} \to H_{2n-1}(\widetilde{C}, \mathbf{Z}) \oplus \pi_{2n-1}((S^n \vee S^n) \otimes \mathbf{Q})$$

is injective. The action of $C(f')_* \otimes \mathbf{Q}$ is obtained by restricting the action of $\widetilde{C(f')}_* \oplus (\rho \otimes \mathbf{Q})_*$ to the image of ν.

We have an isomorphism (see [W] XI (1.6), (1.7))

$$\pi_{2n-1}(S^n \vee S^n) \cong \pi_{2n}(S^n \times S^n) \oplus \pi_{2n-1}(S^n) \oplus \pi_{2n-1}(S^n),$$

and ρ_* acts by $(\rho' \times \rho'')_*$ on the first summand, and by $\rho'_* = \rho''_*$ on the other two summands. Since ρ', ρ'' have degree d^k, one easily computes from Scholium 5 that $(\rho \otimes \mathbf{Q})_*$ acts by $d^{2k} = d^n$ on $\pi_{2n-1}((S^n \vee S^n) \otimes \mathbf{Q})$.

Finally, we have an isomorphism $H_{2n-1}(\widetilde{C}, \mathbf{Z}) \cong H_{2n-2}(C, H_1(S^1))$, so that $\widetilde{C(f')}_*$ acts on $H_{2n-1}(\widetilde{C}, \mathbf{Z})$ by d^n. This completes the proof of (i). $\square$

We now easily complete the proof of the Theorem. Assume by induction that, for all $d \equiv 0 \pmod{m'}$, the map $F_{d,n-2} : Q'^{(n)} \to Q'^{(n)}$ extends to a map $f' : Q' \to Q'$. Let $m = (m'N)^2$, where N annihilates $\pi_{2n-1}(C)_{tors}$. If $d \equiv 0 \pmod{m}$, then $d = d_1 d_2$ where $d_1 = em'N$ and $d_2 = m'N$ for some integer e. We then have self maps f'_1, f'_2 extending $F_{d_1,n-2}$, $F_{d_2,n-2}$ respectively. Then $C(f'_1 \circ f'_2) = C(f'_1) \circ C(f'_2)$ is an extension of $F_{d,n}$, and one readily computes from the above lemma that it acts by multiplication by d^n on $\pi_{2n-1}(C)$. Hence $C(f'_1 \circ f'_2)$ extends to a self map of Q. $\square$

References

[L] R. Lazarsfeld *Some applications of the theory of Positive vector bundles* in *Complete Intersections, C.I.M.E., Acireale 1983.* Lect. Notes. Math. 1092, Springer-Verlag, Berlin, Heidelberg, New York (1984).

[M] S. Mori *Projective manifolds with ample tangent bundles*, Ann. Math. 110 (1979) 593-606.

[KS] K. H. Paranjape and V. Srinivas *Self maps of homogeneous spaces.* preprint.

[W] G. W. Whitehead *Elements of Homotopy Theory*, Grad. Texts in Math. 61, Springer-Verlag, Berlin, Heidelberg, New York (1978).