Algebraic Cycles

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Introduction

This article is based on a talk given by V. Srinivas at the MRI, Allahabad. We give an account of the theory of algebraic cycles where the stress is not on the spate of conjectures (Hodge, Tate, Grothendieck, Bloch-Beilinson, etc.) that define the picture of this theory today, but rather on the key examples that refined and delineated this picture. Some of the deepest aspects of the theory of algebraic cycles are related to number theory. However, because of our lack of expertise on this topic, and to limit the scope of the discussion, we concentrate on the geometric aspects of the theory.

The model case of the theory of divisors on an algebraic curve (or compact Riemann surface) is dealt with in Section 1. The attempt to generalise this theory in higher dimensions is what led to much of the later work. In Section 2 we introduce the Chow ring with its relation to Grothendieck's K-theory via the Grothendieck Riemann-Roch theorem. The theory of divisors on smooth projective varieties is the next best understood case and we describe its features in Section 3. A much studied case is that of zero-cycles which we discuss in Section 4. Various alternate equivalences were introduced on the group of algebraic cycles; we study these in Section 5. We carry on in this section with a survey of the examples that build up our picture of the relation between these equivalences. In Section 6 we see the results of the attempt to relate the Chow group to points on an Abelian variety. As mentioned above we do not survey the conjectures in the theory of algebraic cycles. We discuss some of these briefly in Section 7. More detailed accounts of these can be found in [12] and an excellent survey by U. Jannsen [20].

General references for the theory of algebraic cycles are the survey article [19], describing the status of the subject in the early 70's, and the book [4] giving subsequent developments and newer viewpoints. The material in Section 1 can be found in most books on curves, for example [17]. The book [14] is the most complete source for the construction of the Chow ring, Chern classes, the Grothendieck–Riemann–Roch theorem and other material in Section 2. The theory of divisors on surfaces is dealt with in detail in [25] and a similar treatment can also be given in higher dimensions. References for the remaining sections can be found within the text. Unfortunately no books exist which cover the developments in the theory of algebraic cycles after Bloch's monograph [4] on the subject in 1980.

1 Model case of curves

The topic of algebraic cycles has its origin in the theory of divisors on an algebraic curve, or compact Riemann surface. If X is a non-singular projective curve over an algebraically closed field k, a *divisor* on X is an element of the free abelian group on the points of X; we denote this free abelian group by $\mathrm{Div}\,(X)$. If f is a rational function on X, we can associate to it its divisor $\mathrm{div}\,(f)_X = Z_0(f) - Z_\infty(f)$, where $Z_0(f)$ is the set of zeroes of f, and $Z_\infty(f)$ the set of poles of f, both counted with multiplicity. Such a divisor is called a *principal* divisor; we denote by $P(X)$ the subgroup of $\mathrm{Div}\,(X)$ consisting of principal divisors, and we define the *(divisor) class group* $\mathrm{Cl}\,(X) = \mathrm{Div}\,(X)/P(X)$.

Let $\mathrm{Pic}\,(X)$ denote the group of line bundles (*i.e.*, invertible sheaves) on X. To any meromorphic section of a line bundle L we can associate a divisor in a manner analogous to that for meromorphic functions given above. The divisor associated with a *holomorphic* section of a line bundle is said to be an *effective divisor*; this is equivalent to the assertion that all the multiplicities of points occuring in the divisor are non-negative. The ratio of any two mermorphic sections of L is a global meromorphic function. Thus there is a natural homomorphism $\mathrm{Pic}\,(X) \to \mathrm{Cl}\,(X)$. This map is an isomorphism. By abuse of notation we will denote the divisor class of a line bundle L by L also.

There is a homomorphism $\deg : \mathrm{Cl}\,(X) \to \mathbb{Z}$ called the *degree* homomorphism given by $\deg(\sum_i n_i[P_i]) = \sum_i n_i$. The *Riemann-Roch theorem* states that if L is any line bundle on X,

$$\begin{aligned}\dim_k \mathrm{H}^0(X, L) &= \deg(D) + 1 - g + \dim_k \mathrm{H}^1(X, L)\\ &= \deg(D) + 1 - g + \dim_k \mathrm{H}^0(X, \Omega^1_{X/k} \otimes L^{-1})\end{aligned}$$

Here g is the genus of X and $\Omega^1_{X/k}$ is the invertible sheaf of differential 1-forms. As a consequence one easily obtains the identities $g = \dim_k \mathrm{H}^0(X, \Omega^1_{X/k})$ and $\deg(\Omega^1_{X/k}) = 2g - 2$. Moreover, from the fact that $\mathrm{H}^0(X, L)$ is zero if $\deg(L) < 0$ we obtain that

$$\mathrm{H}^0(X, L) = \deg(D) + 1 - g \quad \text{if } \deg(D) \geq 2g - 1.$$

The collection of all effective divisors of a fixed degree d form the smooth projective variety $\mathrm{Sym}^d(X)$ (the d-th symmetric product of X with itself). The kernel $\mathrm{Cl}^0(X)$ of $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$ is also naturally isomorphic to (the group of k-rational points of) an Abelian variety, the *Jacobian variety* $\mathrm{Jac}(X)$. Fixing a point p_0 on the curve we have a natural morphism $\phi_d : \mathrm{Sym}^d(X) \to \mathrm{Jac}(X)$ sending an effective divisor D to the class of $D - d \cdot p_0$. The *Abel-Jacobi theorem* (which yields the above isomorphism between $\mathrm{Cl}^0(X)$ and $\mathrm{Jac}(X)$) says that the fibre of ϕ_d through a divisor D precisely consists of all effective divisors in the same divisor class as D. Moreover, from the Riemann-Roch theorem we see that ϕ_d is surjective for $d \geq g$.

2 The Grothendieck–Riemann–Roch theorem

Let X be a non-singular variety over k. An *algebraic cycle* of codimension p is an element of the free Abelian group on irreducible subvarieties of X of codimension p; the group of these cycles is denoted $Z^p(X)$. As in the case of curves one can introduce the *effective cycles* $Z^p(X)^{\geq 0}$ which is the sub-semigroup of $Z^p(X)$ consisting of non-negative linear combinations. There is a subgroup $R^p(X) \subset Z^p(X)$, defined to be the subgroup generated by all the cycles $\mathrm{div}(f)_W$ where W ranges over irreducible subvarieties of codimension $p-1$ in X, and $f \in k(W)^*$. The quotient $\mathrm{CH}^p(X) = Z^p(X)/R^p(X)$ is called the *Chow group* of codimension p cycles on X modulo rational equivalence; if $n = \dim X$ then we use the notation $\mathrm{CH}_p(X) = \mathrm{CH}^{n-p}(X)$. For $p = 1$ and X a smooth projective curve the Chow group $\mathrm{CH}^1(X)$ is precisely the class group $\mathrm{Cl}(X)$ introduced above.

The generalisation of Schubert calculus on the Grassmannians is the *intersection product*

$$\mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathrm{CH}^q(X) \to \mathrm{CH}^{p+q}(X)$$

making $\mathrm{CH}^*(X) = \oplus_p \mathrm{CH}^p(X)$ into an associative, commutative, graded ring, where $\mathrm{CH}^0(X) = \mathbb{Z}$, and $\mathrm{CH}^p(X) = 0$ for $p > \dim X$. The Chow ring is thus an algebraic analogue for the even cohomology ring $\oplus_{i=0}^{n}\mathrm{H}^{2i}(X,\mathbb{Z})$ in topology. A refined version of this analogy is examined in Section 6. In any case we note the following 'cohomology-like' properties.

1. $X \mapsto \oplus_p \mathrm{CH}^p(X)$ is a contravariant functor from the category of smooth varieties over k to graded rings.

2. If X is projective and $n = \dim X$, there is a well defined *degree homomorphism* $\deg : \mathrm{CH}^n(X) \to \mathbb{Z}$ given by $\deg(\sum_i n_i P_i) = \sum_i n_i$. This allows one to define intersection numbers of cycles of complementary dimension, in a purely algebraic way, which agree with those defined via topology when $k = \mathbb{C}$ (see item 7 below).

3. If $f : X \to Y$ is a proper morphism of smooth varieties, there are 'Gysin' maps $f_* : \mathrm{CH}^p(X) \to \mathrm{CH}^{p+d}(Y)$ for all p, where $d = \dim Y - \dim X$; here if $p + d < 0$, we define f_* to be 0; the induced map $\oplus_p \mathrm{CH}^p(X) \to \oplus_p \mathrm{CH}^p(Y)$ is $\oplus_p \mathrm{CH}^p(Y)$-linear ('projection formula').

4. $f^* : \mathrm{CH}^*(X) \xrightarrow{\cong} \mathrm{CH}^*(V)$ for any vector bundle $f : V \to X$.

5. If V is a vector bundle (*i.e.*, locally free sheaf) of rank r, then there are Chern classes $c_p(V) \in \mathrm{CH}^p(X)$, such that

   (a) $c_0(V) = 1$,

   (b) $c_p(V) = 0$ for $p > r$, and

   (c) for any exact sequence

   $$0 \to V_1 \to V_2 \to V_3 \to 0$$

   we have $c(V_2) = c(V_1)c(V_3)$, where $c(E_i) = \sum_p c_p(V_i)$ are the total Chern classes.

   Moreover, we also have the following property.

6. If $f : \mathbb{P}(V) \to X$ is the projective bundle associated to a vector bundle of rank r, $\mathrm{CH}^*(\mathbb{P}(V))$ is a $\mathrm{CH}^*(X)$-algebra generated by $\xi = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$, the first Chern class of the tautological line bundle, which is subject to the relation

   $$\xi^r - c_1(V)\xi^{r-1} + \cdots + (-1)^r c_n(V) = 0$$

7. If $k = \mathbb{C}$, there are cycle class homomorphisms $\mathrm{CH}^p(X) \to \mathrm{H}^{2p}(X, \mathbb{Z})$ such that the intersection product corresponds to the cup product in cohomology, and for a vector bundle E, the cycle class of $c_p(E)$ is the topological p-th Chern class of E.

In analogy with the case of curves we have that $c_1 : \mathrm{Pic}\,(X) \to \mathrm{CH}^1(X)$ is an isomorphism. In fact more is true. If $K_0(X)$ is the Grothendieck ring of vector bundles on X, the *Chern character* (defined using Chern classes by the same formula as in topology) gives a ring isomorphism

$$ch : K_0(X) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{CH}^*(X) \otimes \mathbb{Q}.$$

Identifying the group $K_0(X)$ with the Grothendieck group $G_0(X)$ of coherent sheaves, we may extend the definitions of Chern classes and Chern character to coherent sheaves; now the *Grothendieck-Riemann-Roch theorem* states that for any proper morphism $f : X \to Y$, and any coherent sheaf $\mathcal{F}$ on X, we have

$$f_*(ch(\mathcal{F})td(X)) = ch(f_!\mathcal{F})td(Y),$$

where $td(X) \in \mathrm{CH}^*(X)$, $td(Y) \in \mathrm{CH}^*(Y)$ are the Todd classes of the tangent sheaves of X and Y respectively; here $f_! : G_0(X) \to G_0(Y)$ is $f_!(\mathcal{F}) = \sum_{i\geq 0}(-1)^i[R^if_*\mathcal{F}]$, and the Todd class of a coherent sheaf is a certain polynomail in its Chern classes. If X is proper over k (*e.g.*, X is projective) of dimension n, and Y is a point, this gives a formula (the Grothendieck-Hirzebruch-Riemann-Roch formula)

$$\chi(X, \mathcal{F}) = \sum_{i\geq 0}(-1)^i \dim_k \mathrm{H}^i(X, \mathcal{F}) = \deg\left(ch(\mathcal{F})td(X)\right)_n,$$

where the subscript n means that we compute the degree of the component in $\mathrm{CH}^n(X)$. For further details, see [14], Chapter 15.

3 Divisors on varieties of higher dimension

For the special case of divisors (*i.e.*, $\mathrm{CH}^1(X)$) much of the picture is unchanged from that for curves. To begin with, as we saw above we have an isomorphism $\mathrm{Pic}\,(X) \cong \mathrm{CH}^1(X)$.

When X is projective one can define an equivalence relation on $\mathrm{CH}^1(X)$ as follows. Let C be any smooth curve and $D \subset X \times C$ be a divisor which does not contain any fibre of $X \times C \to C$. For any pair of points p,q on C the divisor $D \cap X \times \{p\} - D \cap X \times \{q\}$ can be considered as a divisor on X, which is said to be *algebraically equivalent* to 0. The quotient of $\mathrm{CH}^1(X)$ by this equivalence relation is a finitely generated Abelian group—the *Néron-Severi group* $\mathrm{NS}(X)$. This gives us a generalisation of the degree homomorphism for curves, namely the quotient map $cl : \mathrm{CH}^1(X) \to \mathrm{NS}(X)$.

Upto torsion this equivalence relation can also be defined using intersection theory. We define a divisor D to be *numerically equivalent* to zero if the intersection number $(D \cdot C) = 0$ for every curve C contained in X. Then one knows that some multiple of D is in fact algebraically equivalent to 0. Conversely, if a divisor D is algebraically equivalent to 0 then it is also numerically equivalent to 0. In case the ground field is $\mathbb{C}$ then we can also identify algebraic equivalence with homological equivalence: *i.e.*, a divisor is algebraically equivalent to 0 precisely if it lies in the kernel of the cycle class map $\mathrm{CH}^1(X) \to \mathrm{H}^2(X, \mathbb{Z})$.

In particular, $\deg(ch(L) \cdot td(X))_n$ depends only on the class $cl(L)$ of L in $\mathrm{NS}(X)$. The Grothendieck-Hirzebruch-Riemann-Roch theorem then actually gives a method for computing $\chi(X, L)$ in terms of the class $cl(L)$ in $\mathrm{NS}(X)$. However, the exact formula $\dim_k \mathrm{H}^0(X, \mathcal{O}_X(D)) = \deg D + 1 - g$, valid for divisors of large degree on a curve of genus g, has only a partial generalisation to higher dimensions: if D is an *ample* divisor, then the Grothendieck-Riemann-Roch theorem gives a formula for $\dim_k \mathrm{H}^0(X, \mathcal{O}_X(mD))$ for large m, since $\mathrm{H}^i(X, \mathcal{O}_X(mD)) = 0$ for $i > 0$ (by Serre's vanishing theorem), and so

$$\dim_k \mathrm{H}^0(X, \mathcal{O}_X(mD)) = \chi(\mathcal{O}_X(mD)) = deg(ch(L)td(X))_n \ .$$

For effective divisors D on a surface, the general case was studied by Zariski[43], and its solution is completed in [11], where a similar problem is posed for suitable divisors on varieties of dimension ≥ 3.

The collection of all effective divisors on X corresponding to a fixed class c in $\mathrm{NS}(X)$ form a projective scheme $Hilb_c(X)$. Also, in analogy with the case for curves, the kernel $A^1(X)$ of the morphism $\mathrm{CH}^1(X) \to \mathrm{NS}(X)$ is also naturally isomorphic to (the group of k-rational points of) an Abelian variety, the *Picard variety* $\mathrm{Pic}^0(X)$. Fixing one divisor C in the class c we obtain a natural morphism $Hilb_c(X) \to \mathrm{Pic}^0(X)$, the *Abel-Jacobi morphism.* The fibres of this morphism precisely consist of effective divisors corresponding to a fixed class in $\mathrm{CH}^1(X)$. As in the case of curves, one can show that for a "sufficiently large" multiple of an ample class c the morphism $Hilb_{m \cdot c} \to \mathrm{Pic}^0(X)$ is surjective.

4 Zero cycles on varieties of higher dimension

Another way of looking at divisors on curves is as zero dimensional cycles. For a higher dimensional X we now examine $\mathrm{CH}_0(X)$. Let $\dim X = n$. We put $\mathrm{CH}_0(X)_0 = \ker(\deg : \mathrm{CH}^n(X) \to \mathbb{Z})$, the group of zero cycles of degree 0 (modulo rational equivalence). This conicides with cycles numerically equivalent to zero, and also with cycles algebraically equivalent to zero (as defined in the next section).

There is a surjective regular homomorphism $\mathrm{CH}_0(X)_0 \to \mathrm{Alb}\,(X)(k)$, where $\mathrm{Alb}\,(X)$ is an Abelian variety, the *Albanese variety* of X. An algebraic construction of $\mathrm{Alb}\,(X)$ is as follows. There is a *universal* line bundle P on $X \times \mathrm{Pic}^0(X)$ called the Poincaré bundle. By duality this induces a morphism $X \to \mathrm{Alb}\,(X)$, where $\mathrm{Alb}\,(X)$ denotes the dual Abelian variety to $\mathrm{Pic}^0(X)$. Thus we obtain a morphism $\mathrm{Sym}^d(X) \to \mathrm{Alb}\,(X)$ by additivity. As a complex torus,

$$\mathrm{Alb}\,(X) \cong \mathrm{H}^{2n-1}(X,\mathbb{C})/\left(F^n\mathrm{H}^{2n-1}(X,\mathbb{C}) + \mathrm{image}\ \mathrm{H}^{2n-1}(X,\mathbb{Z})\right),$$

where F^* is the *Hodge filtration* on cohomology.

Now it is not hard to show that for sufficiently large d, the morphism $\mathrm{Sym}^d(X) \to \mathrm{Alb}\,(X)$ is surjective. However, the fibres of this map are *not* in general rational equivalence classes of effective zero cycles of degree d. It is true that $\mathrm{CH}_0(X)_0 \to \mathrm{Alb}\,(X)(k)$ is an isomorphism on torsion subgroups (Roitman's theorem; see [33]). However, if $\mathrm{H}^0(X, \Omega^i_{X/\mathbb{C}}) = \mathrm{H}^{i,0}(X) \neq 0$ for some $i \geq 2$, then $\mathrm{CH}_0(X)_0 \to \mathrm{Alb}\,(X)(k)$ is *not* an isomorphism; in fact $\mathrm{CH}_0(X)_0$ is not the group of points of an Abelian variety in any natural way (this is a result of Mumford [26] for surfaces, generalised to arbitrary dimension by Roitman [32]).

In this situation, a well known conjecture of Bloch asserts that if X is a surface with $\mathrm{H}^{2,0}(X) = 0$, then in fact $\mathrm{CH}_0(X)_0 \cong \mathrm{Alb}\,(X)$. If this is the case, the natural map $\mathrm{CH}^1(C) \to \mathrm{CH}^2(X)$ is surjective, for C any hyperplane section of X (or more generally, an ample divisor). This may be generalized as follows:

(Generalised Bloch's Conjecture) If $\mathrm{H}^{i,0} = 0$ for all $i \geq r$, then there is a subvariety $f : Z \hookrightarrow X$, where $\dim Z = r$, such that $f_* : \mathrm{CH}^r(Z) \to \mathrm{CH}^n(X)$ is surjective; in fact one may expect this to hold for any 'sufficiently ample' Z.

Some examples are known in support of these conjectures; for example, Bloch, Kas and Lieberman [6] showed that Bloch's conjecture (for surfaces) is true for surfaces which are not of general type. Other (rather special) examples have been given by several authors; most recently Voisin [41] has shown that the conjecture holds for Godeaux surfaces. In higher dimensions, Roitman [33] shows that $\mathrm{CH}^n(X) = \mathbb{Z}$ for smooth projective complete intersections with $\mathrm{H}^{n,0}(X) = 0$ (complete intersections always have $\mathrm{H}^{i,0}(X) = 0$ for $i < n$). In [8], it is shown that if X is the (desingularized) Kummer variety associated to an Abelian variety of odd dimension n, then there is a divisor $D \subset X$ such that $\mathrm{CH}^{n-1}(D) \twoheadrightarrow \mathrm{CH}^n(X)$; here $\mathrm{H}^{n,0}(X) = 0$ but $\mathrm{H}^{n-1,0}(X) \neq 0$.

The results of Mumford-Roitman on non-triviality of Chow groups of 0-cycles are over $\mathbb{C}$, or rather, over universal domains; if X is defined over a field k, the above proofs (or variations of them) can be adapted to work over the algebraic closure of the function field $k(X)$ of X over k. This raises the question as to whether the Chow group of 0-cycles is trivial in those cases over smaller algebraically closed fields. Schoen and Nori (see [36]) have constructed examples of surfaces over $\overline{\mathbb{Q}}$ such that over $\overline{\mathbb{Q}(t)}$, an algebraically closed field of transcendence degree 1, the Chow group of 0-cycles of degree 0 differs from the Albanese variety. Conjecturally, for any smooth projective surface over $\overline{\mathbb{Q}}$, the Chow group of 0-cycles of degree 0 is isomorphic to the Albanese; this is a particular instance of the Bloch-Beilinson conjectures. No non-trivial example of this conjecture has been verified, at present.

The above theory for zero-dimensional cycles admits generalizations to the case of singular projective varieties as well; see [38], [39], [37], [23].

Another area of application of the theory of zero cycles is when X is non-projective or even affine. The group $\mathrm{CH}^n(X)$ need not be 0 (unlike the top cohomology $\mathrm{H}^{2n}(X, \mathbb{Z})$), in this case. For example, it is standard to use non-vanishing intersection numbers to provide obstructions to the existence of embeddings in $\mathbb{P}^{2n}_k$ of smooth projective varieties of dimension n; similar arguments can be given for affine varieties of dimension n if the analogous obstruction element in $\mathrm{CH}^n(X)$ is non-zero. Thus the theory of algebraic cycles has applications to the study of projective modules, and to affine algebraic geometry (see [7]). However, these results are usually much subtler than the analogous ones using intersection numbers.

5 Equivalence relations

We have introduced three equivalence relations in the previous two sections, which can be defined as follows for cycles in every codimension.

1. (Algebraic equivalence) $\alpha \in \mathrm{CH}^p(X)$ is *algebraically equivalent to 0* if there is a non-singular projective curve C, an element $\beta \in \mathrm{CH}^p(X \times C)$ and points $x_1, x_2 \in C$ such that $i_1^*\beta - i_2^*\beta = \alpha$, where $i_j : X \hookrightarrow X \times C$ is $i_j(x) = (x, x_j)$. Then β in fact determines a homomorphism $\mathrm{Cl}^0(C) \to \mathrm{CH}^p(X)$, where $Cl^0(C)$ is isomorphic to the group of points of the Jacobian variety of C. The subset of $\mathrm{CH}^p(X)$ of elements algebraically equivalent to 0 is a subgroup $\mathrm{CH}^p_{alg}(X)$; from the definitions, one sees that $\mathrm{CH}^p_{alg}(X)$ is a quotient of a direct sum of Jacobians of projective smooth curves, hence is a divisible Abelian group.
2. (Homological equivalence) If $k = \mathbb{C}$, then $\mathrm{CH}^p_{hom}(X)$ is the kernel of the cycle class homomorphism $\mathrm{CH}^p(X) \to \mathrm{H}^{2p}(X, \mathbb{Z})$.
3. (Numerical equivalence) If X is projective, then $\alpha \in \mathrm{CH}^p(X)$ is *numerically equivalent* to 0 if for any $\beta \in \mathrm{CH}^{n-p}(X)$, the intersection product $\alpha\beta \in \mathrm{CH}^n(X)$ has degree 0. The elements numerically equivalent to 0 form a subgroup $\mathrm{CH}^p_{num}(X) \subset \mathrm{CH}^p(X)$.

For simplicity, we restrict ourselves to the case $k = \mathbb{C}$. Now

$$\mathrm{CH}^p_{alg}(X) \subset \mathrm{CH}^p_{hom}(X) \subset \mathrm{CH}^p_{num}(X) \subset \mathrm{CH}^p(X).$$

One of Grothendieck's *standard conjectures* asserts that

$$\mathrm{CH}^p_{num}(X) = \{x \in \mathrm{CH}^p(X) \mid nx \in \mathrm{CH}^p_{hom}(X) \text{ for some positive integer } n\}.$$

Equivalently,

$$\mathrm{CH}^p_{hom}(X) \otimes \mathbb{Q} = \mathrm{CH}^p_{num}(X) \otimes \mathbb{Q}.$$

One way of proving this could be to attempt to show that $\mathrm{CH}^p_{alg}(X) \otimes \mathbb{Q} \stackrel{?}{=} \mathrm{CH}^p_{num}(X) \otimes \mathbb{Q}$. This holds for divisors and zero cycles in particular. So we introduce the quotient $\mathrm{CH}^p_{hom}(X)/\mathrm{CH}^p_{alg}(X) = \mathrm{Griff}^p(X)$ which is called the pth *Griffiths group* of X; note that $\mathrm{Griff}^n(X) = \mathrm{Griff}^1(X) = 0$ as remarked in the previous sections.

The terminology is because of the famous example of Griffiths [16] showing that $\mathrm{Griff}^2(X) \neq 0$ for a general hypersurface X of degree 5 in $\mathbb{P}^4_{\mathbb{C}}$, and in fact $\operatorname{Griff}^2(X)$ has an element of infinite order. Later, Clemens [10] showed that $\operatorname{Griff}^2(X)$ has infinite rank in this case. Other examples of the non-triviality of the Griffiths group were given by Ceresa [9], who showed that if C is a generic curve of genus ≥ 3 embedded in its Jacobian variety X, and $i(C)$ is the image under multiplication by -1 on X, then $[C] - [i(C)]$ gives an element of infinite order in $\operatorname{Griff}^{g-1}(X)$; for $g = 3$, Nori [28] noted that using the action of Hecke correspondences, this in fact implies that $\operatorname{Griff}^2(X)$ has infinite rank in that case. Further examples of non-triviality or infinite dimensionality of $\operatorname{Griff}^p(X) \otimes \mathbb{Q}$ were obtained by Bardelli [3], Voisin [42] and Paranjape [30].

B. Harris [18] showed that if C is the Fermat quartic curve $U^4+V^4+W^4 = 0$ in $\mathbb{P}^2_{\mathbb{C}}$, then Ceresa's cycle $[C] - [i(C)]$ is non-trivial in $\operatorname{Griff}^2(X)$, where X is the Jacobian of C, by reducing this via iterated integrals to the observation that

$$\frac{2\int_0^1 \left[\int_0^x \frac{dt}{(1-t^4)^{1/2}}\right] \frac{dx}{(1-x^4)^{3/4}}}{\left[\int_0^1 \frac{dt}{(1-t^4)^{1/2}}\right]\left[\int_0^1 \frac{dt}{(1-t^4)^{3/4}}\right]}$$

is not an integer! If this number is irrational, his method implies this element has infinite order in $\operatorname{Griff}^2(X)$, a fact which was proved by other methods by Bloch [4]. This gives an example of such a cycle defined over $\mathbb{Q}$. Schoen [35] showed that for a certain elliptic modular 3-fold X over $\overline{\mathbb{Q}}$ (the field of algebraic numbers), $\operatorname{Griff}^2(X)$ has infinite rank.

In all of these examples, the ambient variety has trivial canonical bundle (tangent bundle, in Ceresa's situation), and one uses image of the cycle under the Abel-Jacobi homomorphism to the intermediate Jacobian of Griffiths' (explained in Section 6). For example, in B. Harris' example, the number whose non-integrality is asserted is essentially an integral of holomorphic 3-form (an element of $\mathrm{H}^{3,0}(X)$), whose value is not a period of that 3-form.

There is a new class of examples of non-triviality of $\operatorname{Griff}^p(X) \otimes \mathbb{Q}$ constructed by M. Nori [29], in which the canonical bundle of the variety is ample, and the intermediate Jacobian in question is 0. Nori has introduced a filtration of the Griffiths' group and one can show that every associated graded term in this filtration can be non-zero (Albano and Collino [1] have shown that it can even be of infinite rank). We discuss this further below in the context of conjectural Lefschetz theorems for Chow groups.

Bloch had asked if the Griffiths group is always divisible (for varieties over algebraically closed fields); very recently, Bloch and Esnault have found a counter-example [5]. Schoen [34] has an example (in positive characteristic) of a smooth variety X such that $\mathrm{Griff}^p(X)$ contains a (non-zero) divisible subgroup, for some p.

Other equivalences have been recently introduced and studied on Chow groups with the idea of settling the standard conjectures and also the Bloch conjecture.

6 Chow groups and Abelian varieties

For any smooth projective variety we can form a countable collection of projective schemes $Hilb_n$ (the *Hilbert* or *Chow schemes*) that parametrise effective cycles of codimension p on X. Let $C(X)$ be a subgroup of $\mathrm{CH}^p(X)$. A homomorphism of groups from $C(X)$ to the group of rational points of a group variety A is called *regular* if for any non-singular variety H parametrising cycles on X lying in $C(X)$, the resulting set maps from the set of rational points on H to the set of rational points of A is induced by a morphism of varieties.

In analogy with the picture for divisors one may ask the following questions:

1. Is there a finitely generated group $N(X)$, a surjective map $\xi : \mathrm{CH}^p(X) \to$ $\to N(X)$, and a regular map from $C(X) = \ker \xi$ to the group of points on an Abelian variety A, so that any regular map from $C(X)$ to an Abelian variety factors through A?

2. Does this induce an isomorphism of $\mathrm{CH}^p(X)$ with $N(X) \times A(k)$? (Here $A(k)$ denotes the group of k-rational points of A.)

3. Is there a variety (or scheme) H parametrising effective cycles on X so that the morphism $H \to A$ is surjective? Or at least can we arrange $\mathrm{im}\, H = \mathrm{im}\, \mathrm{CH}^p(X)$?

There is no general geometric construction of the required Abelian variety. There is a complex torus associated to codimension p cycles, defined by Griffiths, which generalizes the Picard and Albanese varieties. This is called the pth *intermediate Jacobian* of X, and is defined by

$$J^p(X) = \frac{\mathrm{H}^{2p-1}(X, \mathbb{C})}{F^p\mathrm{H}^{2p-1}(X, \mathbb{C}) + \mathrm{im}\,\mathrm{H}^{2p-1}(X, \mathbb{Z})}.$$

Here

$$F^p\mathrm{H}^{2p-1}(X, \mathbb{C}) = \oplus_{p' \geq p}\mathrm{H}^{p', 2p-1-p'}(X)$$

is the pth piece of the *Hodge filtration* of $\mathrm{H}^{2p-1}(X, \mathbb{C})$. We can take the map $\xi : \mathrm{CH}^p(X) \to \mathrm{H}^{2p}(X, \mathbb{C})$ and let $C(X) = \mathrm{CH}^p_{hom}(X) = \ker \xi$ as before. There is a map, which Griffiths calls the *Abel-Jacobi map*,

$$\mathrm{CH}^p_{hom}(X) \to J^p(X).$$

However, this does not have as good properties as the Picard and Albanese maps, in general. The image of $\mathrm{CH}^p_{alg}(X)$ is an Abelian subvariety of $J^2(X)$ whose Lie algebra is contained in $\mathrm{H}^{p-1,p}(X) \subset \mathrm{H}^{2p-1}(X, C)/F^p(\mathrm{H}^{2p-1}(X, \mathbb{C})$.

The Griffiths group $\mathrm{Griff}^p(X)$ is always countable, since all effective algebraic cycles of a fixed degree are parametrized by the points of a (possibly reducible) Chow variety of X; taking the union over all degrees, all effective algebraic cycles lie in a countble collection of connected algebraic families, so that $\mathrm{CH}^p(X)/\mathrm{CH}^p_{alg}(X)$ is countable. Hence if $\mathrm{H}^{i,2p-1-i}(X) \neq 0$ for some $i > p$, then the Abel-Jacobi map cannot be surjective. The restriction of the Abel-Jacobi map to $\mathrm{CH}^p_{alg}(X)$ is a regular homomorphism onto the Abelian variety which is its image; conjecturally, this is the universal regular homomorphism, as in the case of the Albanese map. One also expects the Abel-Jacobi map to be injective on torsion, in general. The injectivity of the Abel-Jacobi map on torsion is known for codimension 2 cycles, from work of Merkurjev and Suslin on the K-theory of division algebras, combined with results of Bloch and Ogus; we discuss this below. The universality of the Abel-Jacobi map on $\mathrm{CH}^2_{alg}(X)$ has been proved by Murre [27] using the injectivity on torsion.

7 Relation with cohomology

As we mentioned above, the Chow groups are an analogue for the even cohomology of a smooth projective variety over $\mathbb{C}$. To make this relation more precise we first examine the cycle class map $\mathrm{CH}^p(X) \to \mathrm{H}^{2p}(X, \mathbb{Z})$.

The theorem of Lefschetz on $(1,1)$ classes asserts that the image of the cycle class map $\mathrm{CH}^1(X) \to \mathrm{H}^2(X,\mathbb{Z})$ is precisely the kernel of $\mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathbb{C})/F^1\mathrm{H}^2(X,\mathbb{C})$, where F^* denotes the Hodge filtration. Equivalently, with respect to the *Hodge decomposition* $\mathrm{H}^m(X,\mathbb{C}) = \oplus_{p+q=m}\mathrm{H}^{p,q}(X)$ into spaces $\mathrm{H}^{p,q}$ of harmonic forms of type (p,q), the classes of divisors in $\mathrm{H}^2(X,\mathbb{Z})$ are precisely those classes α such that $\alpha_{\mathbb{C}} \in \mathrm{H}^2(X,\mathbb{C})$ lies in $\mathrm{H}^{1,1}(X)$.

A similar assertion for $\mathrm{CH}^n(X) \to \mathrm{H}^{2n}(X,\mathbb{Z})$ when $n = \dim X$ is obvious since this homomorphism is surjective. Thus one could conjecture that the image of $\mathrm{CH}^p(X) \to \mathrm{H}^{2p}(X,\mathbb{Z})$ consists precisely of the subgroup of Hodge classes

$$\mathrm{Hg}^p(X,\mathbb{Z}) = \ker(\mathrm{H}^{2p}(X,\mathbb{Z}) \to \mathrm{H}^{2p}(X,\mathbb{C})/F^p\mathrm{H}^{2p}(X,\mathbb{C}))$$

However, this is known to be false unless we tensor with $\mathbb{Q}$. Early counterexamples are in [2], and more recently one knows from the work of Kollár, Mori, Miyaoka [22] that a general hypersurface of degree 125 in $\mathbb{P}^4$ cannot contain a cycle of degree 1. The assertion:

$$\mathrm{CH}^p(X) \otimes \mathbb{Q} \to \mathrm{Hg}^p(X,\mathbb{Q}) \text{ is surjective}$$

is the celebrated *Hodge conjecture*. Many special cases are known but there is no general theorem (or even heuristic) in this direction.

The second relation of Chow groups to cohomology is via Griffiths' Abel-Jacobi homomorphism to the points of the Intermediate Jacobian. As we saw with zero cycles, the kernel of this homomorphism can be very large. On the other hand Bloch's conjecture asserts (in the case of zero cycles) that the kernel is torsion (hence zero by Roitman's theorem) in case the Hodge decomposition of $\mathrm{H}^{2n-k}(X,\mathbb{C})$ has no end terms.

Beilinson has generalised this as follows. We define the level filtration $L^p\mathrm{H}^{2p-k}(X,\mathbb{Q})$ as the intersection the kernels of the restriction homomorphisms $\mathrm{H}^{2p-k}(X) \to \mathrm{H}^{2p-k}(Y)$, where Y runs over all subvarieties of X of dimension *less than* p. Let $k(p)$ be the largest integer such that $L^p\mathrm{H}^{2p-k}(X,\mathbb{Q}) \neq \mathrm{H}^{2p-k}(X,\mathbb{Q})$. Then (conjecturally) the complexity of $\mathrm{CH}^p(X) \otimes \mathbb{Q}$ is "measured" by $k(p)$. In particular, he conjectures that (a) if $k(p) = 0$, then $\mathrm{CH}^p_{hom}(X) \otimes \mathbb{Q} = 0$, and (b) if $k(p) = 1$, then the kernel of the Abel-Jacobi homomorphism should be torsion.

For X a complete intersection subvariety of $\mathbb{P}^N$ we have the result of Esnault–Nori–Srinivas [13] which generalises earlier work of Deligne and

Deligne–Dimca, computing the width of the Hodge structures. From these results one can show that for a fixed p and multidegree, and for sufficiently large N (made precise by their results), we have $k(p) = 0$ (if Grothendieck's generalised Hodge conjecture is true). The Chow groups of these varieties ought to be $\mathbb{Z}$ upto torsion. One can see that Roitman's theorem on zero cycles precisely achieves the predicted result. For higher dimensional cycles a weak form of this conjecture is known; see [24], [31].

Another way of examining the consequences of this conjecture is to look at the situation of a smooth subvariety Y of X such that the restriction map on cohomology induces an isomorphism $L^p\mathrm{H}^{2p-k}(X, \mathbb{Q}) \to L^p\mathrm{H}^{2p-k}(Y, \mathbb{Q})$. One such case is again the complete intersection situation, where the Lefschetz hyperplane section theorems give us such isomorphisms. In this case the Grothendieck-Lefschetz theorem precisely achieves the bound predicted. This theorem asserts that the restriction map $CH^1(X) \to CH^1(Y)$ is an isomorphism for $n = \dim Y \geq 3$, and an inclusion for $n = 2$.

A more refined analysis using the monodromy of Lefschetz pencils shows that for $n = 2$, we have $CH^1(X) \cong CH^1(Y)$ provided Y is a *general* hyperplane section; this is called the *Noether-Lefschetz theorem*. Similarly, it is known that for $n = 1$, $CH^1(X) \to CH^1(Y)$ is injective for general Y.

The higher dimensional analogue of the Grothendieck-Lefschetz theorem was posed as a problem by Hartshorne [19] (well before Beilinson formulated his conjectures). Only very weak results are known in this direction [31]. Higher dimensional analogues of the Noether-Lefschetz theorems have recently been formulated by Nori [29] with cohomological justification rather similar to the Beilinson conjectures. Some very weak statements along these lines are known [15], [40], [21].

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