1. Foundations of geometry

Euclidean Geometry is the attempt to build geometry out of the rules of logic combined with some "evident truths" or axioms. The axioms of Euclidean Geometry were *not* correctly written down by Euclid, though no doubt, he did his best. There are now a number of different ways of giving the formal basis for the *same* geometry. These are

1. The "High School Geometry" text book approach.
2. Hilbert's "Foundations of Geometry" approach.
3. Through Projective Geometry as in Coxeters' "Non-Euclidean Geometry".
4. Trough the study of the Euclidean group as done by Sophus Lie.

We shall examine the middle two approaches in the following text. The first method which was learned in school should now be forgotten since we are looking at (to paraphrase Klein) "elementary mathematics from an advanced standpoint".

The method that (to my mind) comes closest to the original approach is that of Hilbert's Foundations of Geometry. Unlike the "High School Geometry" text books, this makes no reference to the "Ruler Placement Postulate" or a "Protractor Placement Postulate", both of which are anti-thetical to a purely geometric approach. The arithmetic aspects of geometry should grow out of it rather than be imposed from outside. Another difference is that the notion of a line is not *as* a set of points in Euclid's approach; points, lines and planes are distinct notions in Hilbert's approach too.

Without much more ado then let us examine Hilbert's axioms for Euclidean geometry. The fundamental notions are points (denoted by A, B, C, ...), lines (denoted by a, b, c, ...) and planes (denoted by α, β, γ, ...). The mutual relations between these are those of Incidence ("contains" or "lies on"), Order ("is between") and Congruence. The axioms characterise the "evident" or fundamental properties of these relations. We divide the axioms into four classes, Incidence, Order, Parallels, Continuity, Congruence.

1.1. **The Axioms of Incidence.**

The following axioms set out the basic incidence relations between lines, points and planes. They also characterise the concept of "dimension" that we associate with these notions.

1. Incidence between points and lines:
   (a) There are at least two distinct points.
   (b) There is one and only one line that contains two distinct points.
   (c) Every line contains at least two distinct points.
2. Incidence between points and planes:
   (a) There are three points that do not all lie on the same line.
   (b) For any three points that do not lie on the same line there is a one and only one plane that contains them.
   (c) Any plane contains at least three points.

3. Incidence between lines and planes:
   (a) If a line lies on a plane then every point contained in the line lies on that plane.
   (b) If a line contains two points which lie on a plane then the line lies on the plane.
4. Dimensionality of space:
   (a) If two planes both contain a point then they also contain a line.
   (b) There are at least four points that do not all lie on the same plane.

The first four axioms (which do not refer to planes) are called the plane geometry axioms, while the remaining are the space axioms. Out of the various Theorems that can be proved we note

Theorem 1. *Given a line and a point not on it there is one and only one plane that contains the line and the point.*

Theorem 2. *Given a pair of lines which meet in a point there is one and only one plane that contains the lines.*

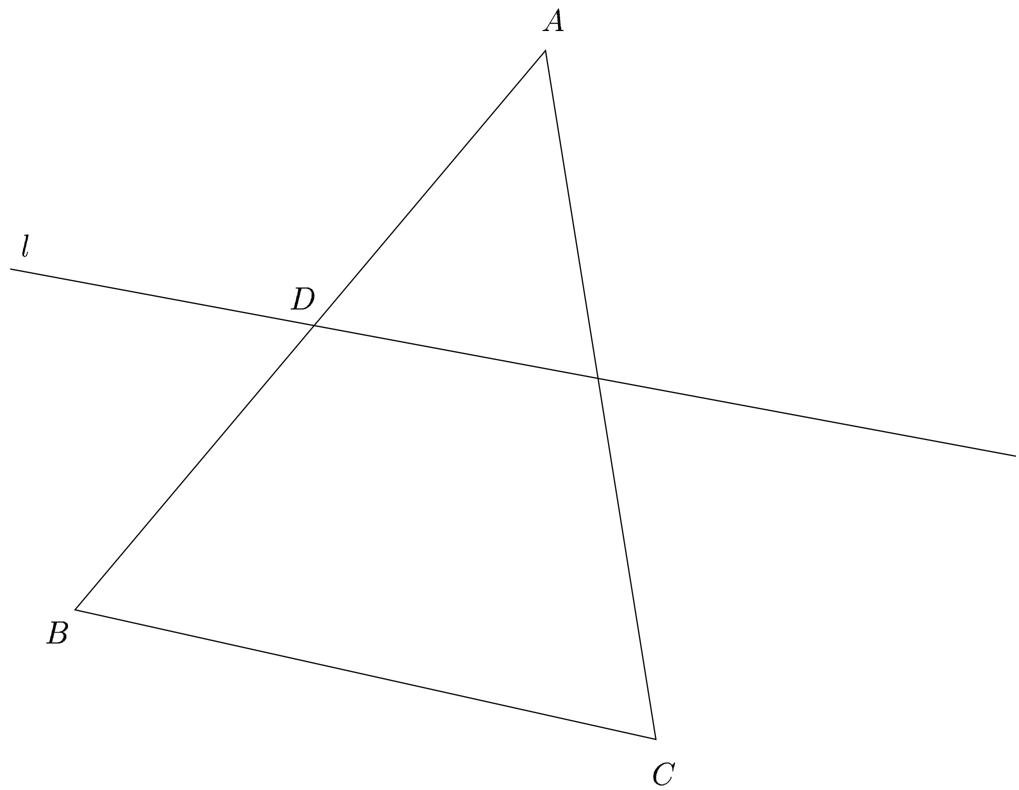
Theorem 3. *Given four points that do not all lie on a plane, there is no line containing three of these points.*

Exercise 1. There is a "geometry" consisting of 4 points, 6 lines and 4 planes that satisfies these axioms.

Exercise 2. Which of the above axioms can be omitted? For those that are necessary construct a "geometry" that satisfies the chosen axiom and defies the others.

1.2. **Axioms of Order.** These axioms were almost ignored by Euclid except the second one below. Their importance was noticed by M. Pasch who saw how they were implicitly being used in many proofs. This is one problem with "evident truths"; we often forget to state some of the axioms and then the geometry is incomplete without them. The following axioms make clear the notion of a point lying between two other points.

1. When B is between A and C then, A, B and C are distinct points lying on a line and B is between C and A.
2. Given a pair of points A and B there is a point C so that B is between A and C.
3. If B lies between A and C then A does not lie between B and C.
4. Let A, B and C be three points on a plane α and a be a line on α that does not contain any one of these points. If there is a point D on a that is between A and B then either a contains a point between A and C or a contains a point between B and C.



The Line l contains a point on one of the other two sides

The first three axioms allow us to introduce the notion of a half-line or ray. Given a pair of points A and B the half ray starting at B and pointing away from A consists of all points C so that B is between A and C. Similarly, the last axiom allows us to introduce the notion of a half plane. Given a point A and a line a the half-plane bounded by a and opposite to A consist of all points B so that a contains a point lying between A and B.

In spite of the axioms of order being ignored for so many hundreds of years they are so important that one can entirely replace the axioms of incidence by giving an extended set of axioms of order. Think of it this way. If a straight line is to be the shortest path from a point to another then we must at least be able to say what are the points "on the way" or in-between.

The following theorems can be deduced from the axioms of Incidence and Order.

Theorem 4. *Given any two point A and B there is a point C that lies between A and B.*

Theorem 5. *Given three points A, B and C that lie one a line exactly one point that lies between the other two.*

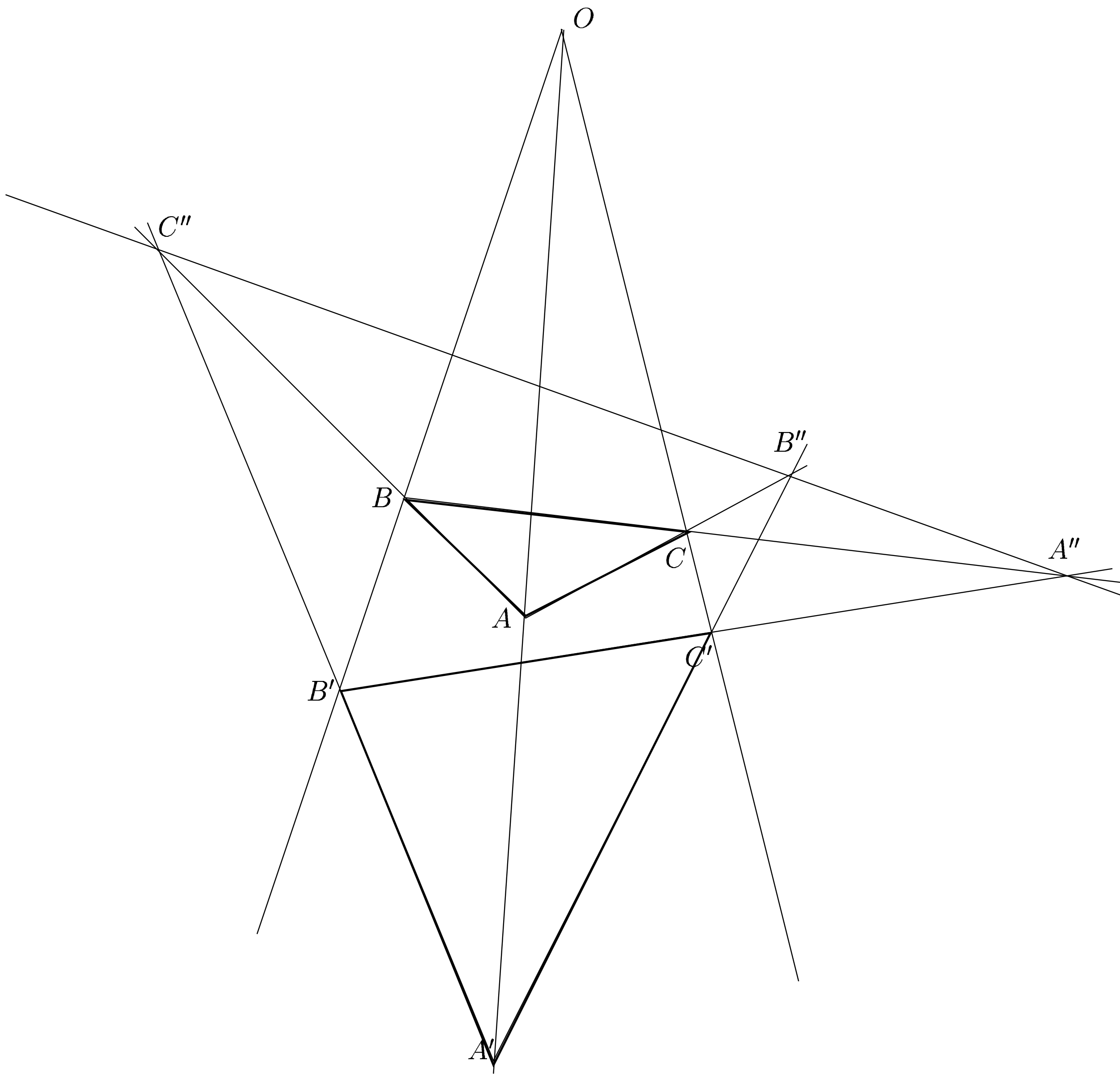
Theorem 6. *Given four points on a line they can be labelled A, B, C and D so that B is between A and C and between A and D and C is between B and D and between A and D.*

Theorem 7. *Given any finite set of points on a line they can be labelled $A_1, A_2, \ldots, A_n$ so that the points are in that order.*

An important theorem that can be deduced from the order axioms was first discovered by G. Desargues:

Theorem 8. *Given points A, B, C, A', B' and C' so that the lines AA', BB' and CC' all pass through a point O. Further, let AB and $A'B'$ meet in a point C'', AC and $A'C'$ meet in a point B'', BC and $B'C'$ meet in a point A''; moreover, let us assume that the 10 points considered are distinct. Then the points A'', B'' and C'' all lie on a line.*

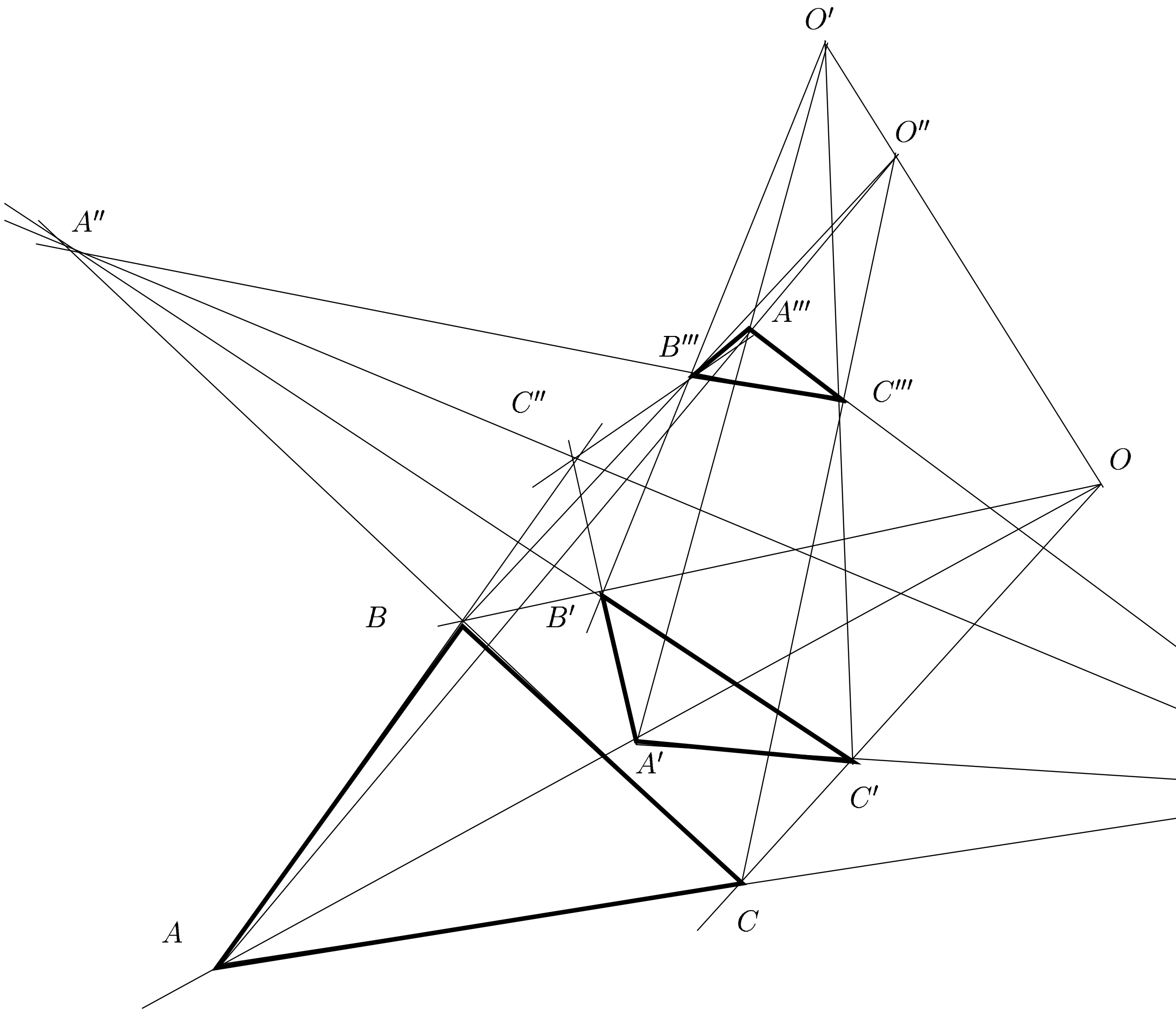
Proof. In case the plane α containing the points A, B and C does not contain all of the points A', B' and C' then the line containing the points A'', B'' and C'' is just the line of intersection of α with the plane α' determined by A', B' and C'. Thus, the theorem needs only to be proved under the assumption that all the points lie in a plane. In this case we shall show how to construct A''', B''' and C''' that do not lie in the plane and so that the points A, B, C, A''', B''' and C''' also satisfy the hypothesis of the theorem. Moreover, A''', B''' and C'' are collinear and so on cyclically. Thus, the planar version will then follow from the non-planar version.



The triangles ABC and $A'B'C'$ lie in different planes.

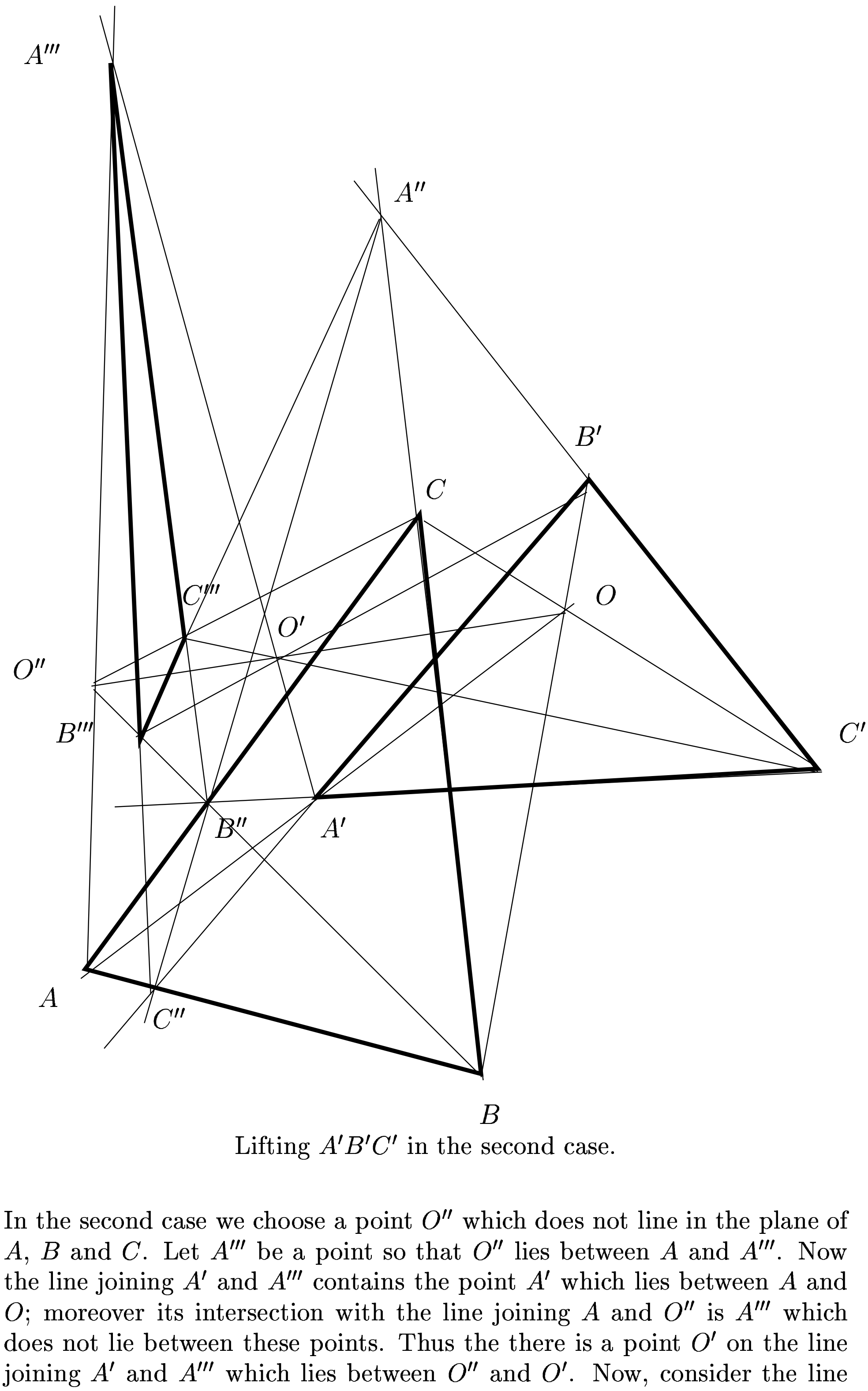
In the remaining cases we examine all the possibilities for between-ness for the triples (A, A', O), (B, B', O) and (C, C', O). By interchanging the $'$ and permuting the letters (A, B, C) we can reduce to the following two cases.

1. A does not lie between A' and O, B does not lie between B' and O, C does not lie between C' and O.
2. A' lies between A and O, B' does not lie between B and O, C' does not lie between C and O.



Lifting $A'B'C'$ in the first case.

Examining the first case, let O' be a point not in the plane of A, B and C. Let A''' be a point between A' and O'. The line joining A and A''' then must contain a point O'' lying between O and O' since its intersection with the line joining A' and O is A which does not lie between these points by hypothesis. Now the line joining O'' and B contains a point O'' between O and O' and thus must contain a point B''' between O' and B' since the point B of intersection of this line with the line joining B' and O is not between these two points. Similarly, we obtain C''' lying on the line joining C and O'' and lying between O' and C'.



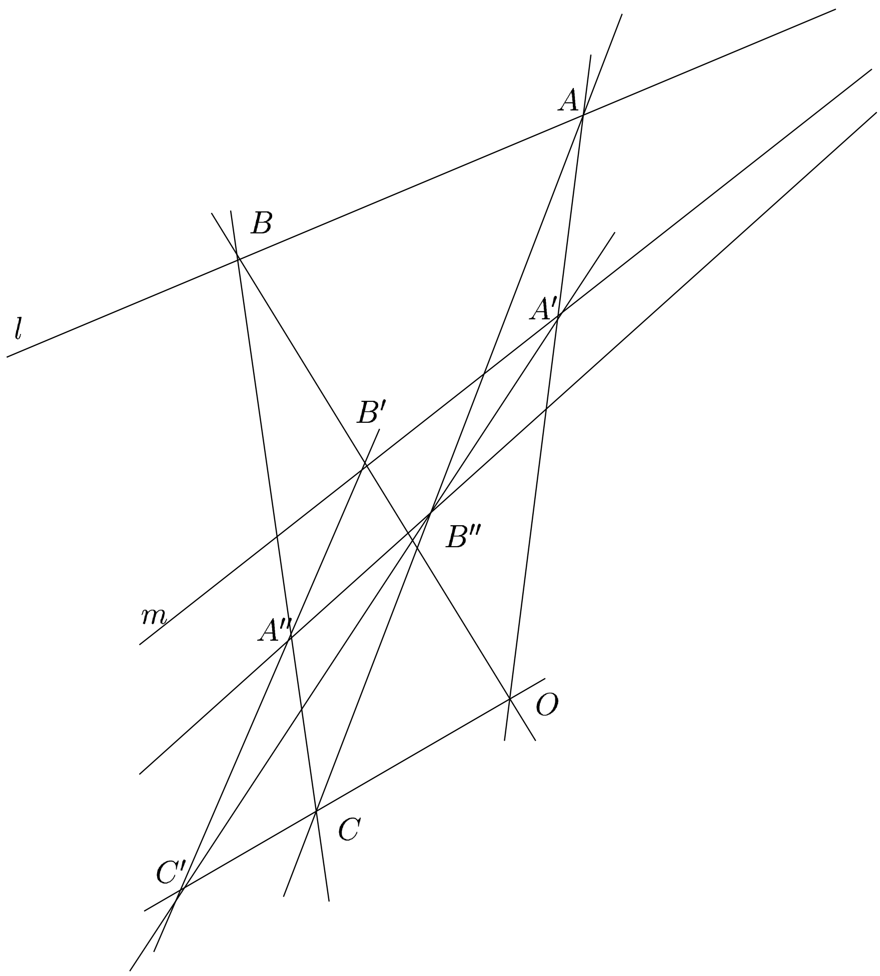
Lifting $A'B'C'$ in the second case.

In the second case we choose a point O'' which does not line in the plane of A, B and C. Let A''' be a point so that O'' lies between A and A'''. Now the line joining A' and A''' contains the point A' which lies between A and O; moreover its intersection with the line joining A and O'' is A''' which does not lie between these points. Thus the there is a point O' on the line joining A' and A''' which lies between O'' and O'. Now, consider the line joining O' and B' and the triangle of points O'', B and O. As before we find a point B''' which lies between B and O'' and on the line joining O' and B'. Similarly, we find C'''.

In both these cases the line joining A''' and B''' meets the plane α within the intersection of the plane determined by A', B' and O' and the plane α; this is the line joining A' and B'. Similarly, the intersection of the line joining A''' and B''' with α lies within the intersection of the plane determined by A, B and O'' with the plane α; this is the line joining A and B. In other words, the line joining A''' and B''' contains the point C''. We prove the other containments cyclically. □

This theorem and the ideas behind its proof allow us to make constructions like:

Exercise 3. Given a pair of non-parallel lines on a piece of paper (in a bounded region of a plane) and a point not on these lines; construct a line through this point that passes through the point of intersection of the two lines (which may not be within the sheet of paper!).



The line joining A'' and B'' passes through the intersection of l and m.

Another aspect of Desargues' theorem is that it's proof makes use of the non-planar axioms of incidence. We shall see that

Theorem 9. *If we have a plane which satisfies all the planar axioms of Incidence and the axioms of order, then it can be embedded in a geometry that satisfies the axioms of space if and only if Desargues' theorem is valid in this plane.*

1.3. **Axiom of Parallels.** The axiom in this section caused the most controversy and confusion of all. The axioms of parallels (which is also an incidence axiom) is

Axiom of Parallels: Given a line and a point outside it there is exactly one line through the given point which lies in the plane of the given line and point so that the two lines do not meet.

Note that, while asserting that there *is* a line through the given point that doesn't meet the given line, it also says there is only one such line. In other words, it also asserts that all the "other" lines co-planar with the given line meet that line. This motivates the introduction of the following (stronger and stranger) version of the Axiom of Parallels:

Projective Axiom of Parallels: Any pair of lines that lie in the same plane meet.

The idea behind this axiom is that even (apparently) parallel lines appear to meet at the horizon. We can demonstrate that this axiom is consistent with the axioms of Incidence by means of Linear Algebra as in the examples below.

1.4. **Models.** One way of checking the consistency of our system of axioms is to construct "models" for which all the axioms are verified. Of course, these verifications again use results from some other area of mathematics and the axioms of that would also have to be verified to be consistent and so on. This is the idea behind the impossibility of verifying consistency. Leaving philosophical studies behind let us examine "three dimensional projective geometry over a (skew-)field".

Exercise 4. Let K be a skew-field (i.e. K has addition, subtraction, multiplication and division but multiplication does not necessarily commute). Points, lines and planes of $P^3(K)$ are given by (left) linear subspaces of K^4 of rank 1, 2 and 3 respectively. The incidence relations are just the inclusions of subspaces. Show that this gives a system that satisfies the above Incidence axioms *and* the projective axiom of parallels.

In fact, this even leads to another system which satisfies the usual axiom of parallels.

Exercise 5. Let $A^3(K)$ be the collection of all points, lines and planes in $P^3(K)$ that are not contained in a fixed plane α (called the plane "at the horizon"). Show that this geometry satisfies all the axioms of incidence and the "usual" axiom of parallels.

The notion of between-ness can also be brought in with some more algebra.

Definition 1. A positivity on K is a subset P so that:

1. $P + P \subset P$ and $P \cdot P \subset P$.
2. $P \cup \{0\} \cup (-P) = K$ and this is a disjoint union.

This conforms to the concept of positive numbers. Using this we can define the cone generated by a collection of vectors in K^4 as the collection of all non-negative linear combinations of the vectors.

Exercise 6. Fix a three dimensional linear subspace V of K^4 (in other words a plane in $P^3(K)$) and a vector v *not* in V. There is a unique linear functional on K^4 which has kernel V and takes the value 1 on v. We say a vector w is *positive* if $f(w)$ lies in P. Every linear subspace in K^4 which does not line in V is then determined by its *positive half.*

Exercise 7. We say that a point A of $A^3(K)$ lies between points B and C if the positive half of the linear subspace in K^4 corresponding to A is a positive linear combination of of the positive halves of the linear subspaces corresponding to B and C respectively. Check that the axioms of order are satisfied on $A^3(K)$ with this notion of between-ness.

We have thus constructed a geometry satisfying all our axioms by making use of some algebra. Other geometries satisfying these axioms can also be constructed.

Definition 2. A collection R of points in $A^3(K)$ is said to be *convex* if, given A and B are points in R and C in $A^3(K)$ is between A and B, then C is also in R.

Definition 3. A convex collection R of points is said to be *open* if for any point A in R and B in $A^3(K)$, there is a point C lying between A and B in $A^3(K)$ so that C is also in R.

Exercise 8. Let R be an open convex collection of points in $A^3(K)$. We denote by $[R]$ the geometry for which points are the points of R, lines and planes of $[R]$ are the lines and planes of $A^3(K)$ which meet R. The relations of incidence and order are inherited from $A^3(K)$. Check that this geometry satisfies the axioms of incidence and order.

A very important result (a sketch of proof is outlined in the next section) is that *every* geometry satisfying the axioms of incidence and order is of the type $[R]$ for an open convex set R in $A^3(K)$ for a suitable ordered field K.

Hence, and this is important to note, the fact that arithmetic/algebraic problems arise in geometry does not immediately have anything to do with measurement! In particular, the relation between distance and coordinates can be much more complicated than that which will emerge from the Pythagoras theorem.

1.5. **Putting co-ordinates.** First of all we expand our geometry and introduce ideal points, lines and planes. This can be considered as the process of re-constructing $P^3(K)$ given [R].

The definition of an ideal point (Point) is motivated by the fact that a pair of intersecting lines is lies in a plane.

Definition 4. A *Point* is a collection of lines so that any pair of lines in this collection is co-planar. Moreover, there is at least one line in this collection through any given point.

The relation of these properties to Desargues' theorem is given by.

Theorem 10. *Let a, b, c be any three lines in the collection of lines determining a Point. There are points A, A' on a, B, B' on b and C, C' on c which give a Desarguian configuration. That is to say, the lines AB and $A'B'$ intersect in a point C'', the lines AC and $A'C'$ intersect in a point B'' and the lines CB and $C'B'$ intersect in a point A'', so that the point A'', B'' and C'' lie on a line.*

Similar ideas can be used to show that there are Points.

Exercise 9. Let l and m be lines and consider the collection of all lines k such that either one of the following holds:

1. k is *not* co-planar with l and m but is co-planar with each of l and m separately.
2. There is a line j of the above type so that j is co-planar with each of k, l, m.

Check that this collection gives a Point.

The notion of collinearity for Points can be defined as follows

Definition 5. The Points A, B and C is said to be *collinear* if for any point p there are lines a, b and c in the collections A, B and C respectively so that a, b and c are coplanar.

With this definition we can define an ideal line

Definition 6. A Line is a collection of Points so that any three are collinear; moreover, the collection contains at least two Points.

We easily check

Exercise 10. There is a unique Line containing a pair of Points.

Similarly, we can define an ideal plane

Definition 7. A Plane is a collection of Points and Lines so that there is at least one Point and one Line not containing it in the collection. If a Line is in the collection then so is every Point contained in the Line. If a Line and a Point not on it are contained in the collection so is every Line containing the Point and a Point of the Line. Finally, there is at least one Point not in the collection.

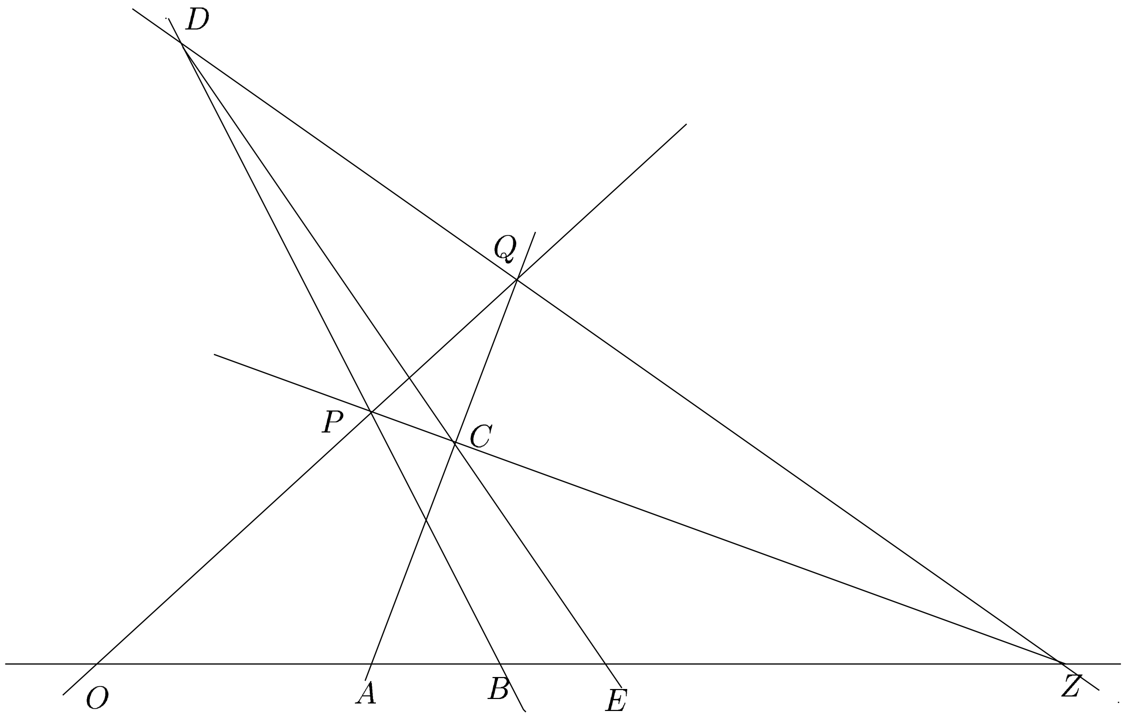
The main result is then

Theorem 11. *If we start with a geometry satisfying the axioms of incidence and order, then the Points, Lines and Planes as defined above satisfy the axioms of incidence and the projective axiom of parallels.*

Since each point, line and plane in the original geometry determines a Point, Line and Plane respectively, we see that we have "embedded" our geometry in a geometry satisfying the axioms of incidence and the projective axiom of parallels. Such a geometry is called an axiomatic projective geometry. By replacing the use of Pasch axiom by the use of the projective axiom of parallels one can also prove Desargues' theorem for this geometry. We wish to co-ordinatise this geometry.

The addition and multiplication operations can be "constructed" in a manner similar to that in Euclidean geometry.

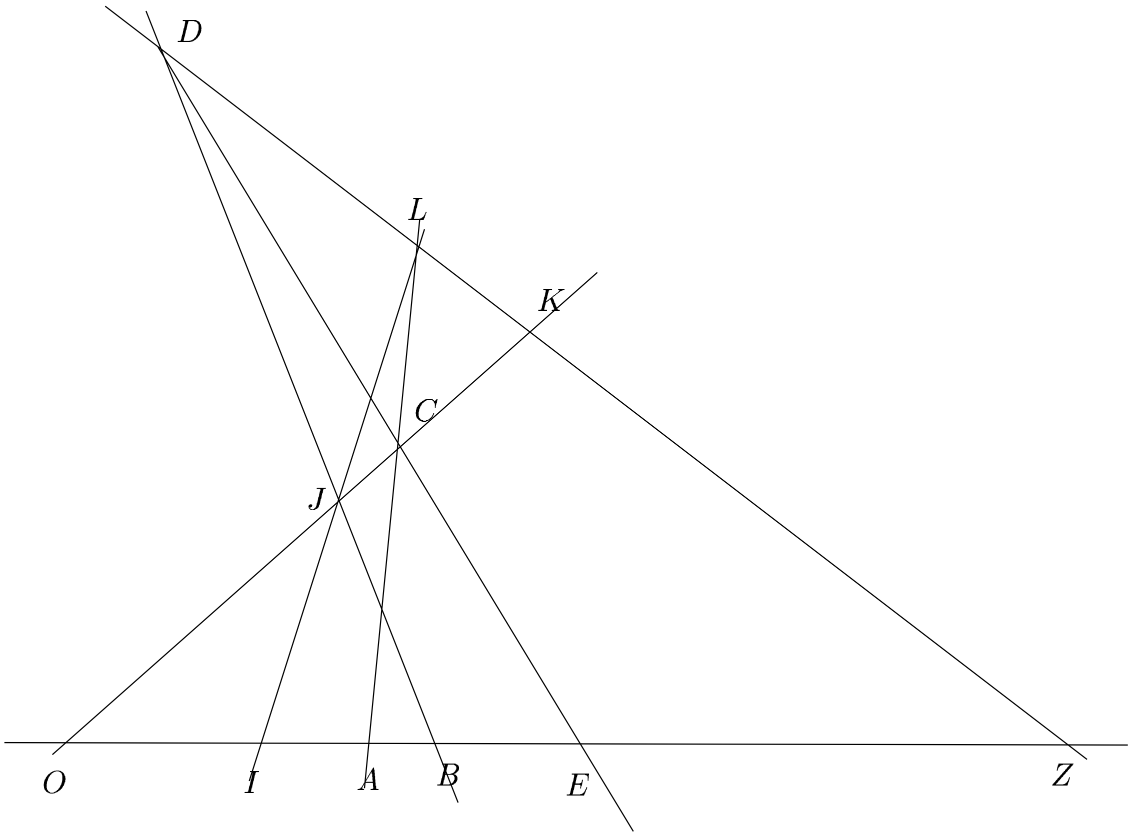
Definition 8. Let O and Z be a pair of Points and A, B be Points on the Line j containing O and Z. Let k be any Line containing Z and not O and P be a point of k other than Z. Let Q be a Point on the Line l joining O and P other than these two. The Line m joining Z and Q meets the Line n joining A and P in a Point C. Let D be the Point of intersection of the Line o joining B and Q. Finally E is the Point of intersection of the line p joining C and D with the line j. We then say that E is the "sum" of the points A and B with respect to O and Z.



The sum of A and B with respect to O and Z is E.

Exercise 11. Show that this operation is well-defined.

Definition 9. Let O, I and Z be distinct collinear Points and A, B be Points on the Line j containing O and Z. Let C be a Point not on j. Let k be the Line joining C with Z and l be the Line joining C with O. Let J be a Point on l different from C and O. The Line m joining I and J meets l in a point K. Let C be the Point of intersection of the Line n joining A and K. Let D be the Point of intersection of the Line joining B and J. Finally, let E be the Point on j that lies on the line o that joins C and D. We say that E us the "product" of the points A and B with respect to O, I and Z.



The product of A and B with respect to O, I and Z is E.

Exercise 12. Show that this operation is well-defined.

The main result is to use these operations to show that

Theorem 12. *Let O, I and Z be distinct Points on a Line j. The operations defined above make the collection of all Points on j other than Z into a (skew-) field K.*

Finally, one produces a one-to-one correspondence

Theorem 13. *There is a one-to-one correspondence between the (axiomatic) projective geometry and the geometry $P^3(K)$.*

The original geometry is then that of a convex set R in $A^3(K)$.

1.6. **Suggested Reading.** Most “advanced” books on Projective Geometry or on Non-Euclidean Geometry will have a good portion of this material. Here are some books that I have found useful.

1. D. Hilbert, *Foundations of Geometry.*
2. H. S. M. Coxeter, *Non-Euclidean Geometry.*
3. Seidenberg, *Projective Geometry.*