1. Coordinate Geometry

The "ab initio" approach to geometry led us in a natural way to coordinate geometry. Thus algebra (and arithmetic) play a very important role in geometry. Algebra is very useful since we can discuss curves and other objects with much greater ease using coordinates. This is not to say that everything in geometry cannot be discussed without bringing in algebra. One can discuss "loci"; each locus is thought of as a geometric figure "traced" by a point moving according to some specific geometric constraint. We will discuss how conics can be described in this way. Though this description is beautiful, it also shows how limited we are if we ignore algebraic techniques. Thus we shall bring in some algebra to study curves and then surfaces. In particular, we shall discuss whether curves are really curved and the curvature of a surfaces is superficial!

1.1. Conics.

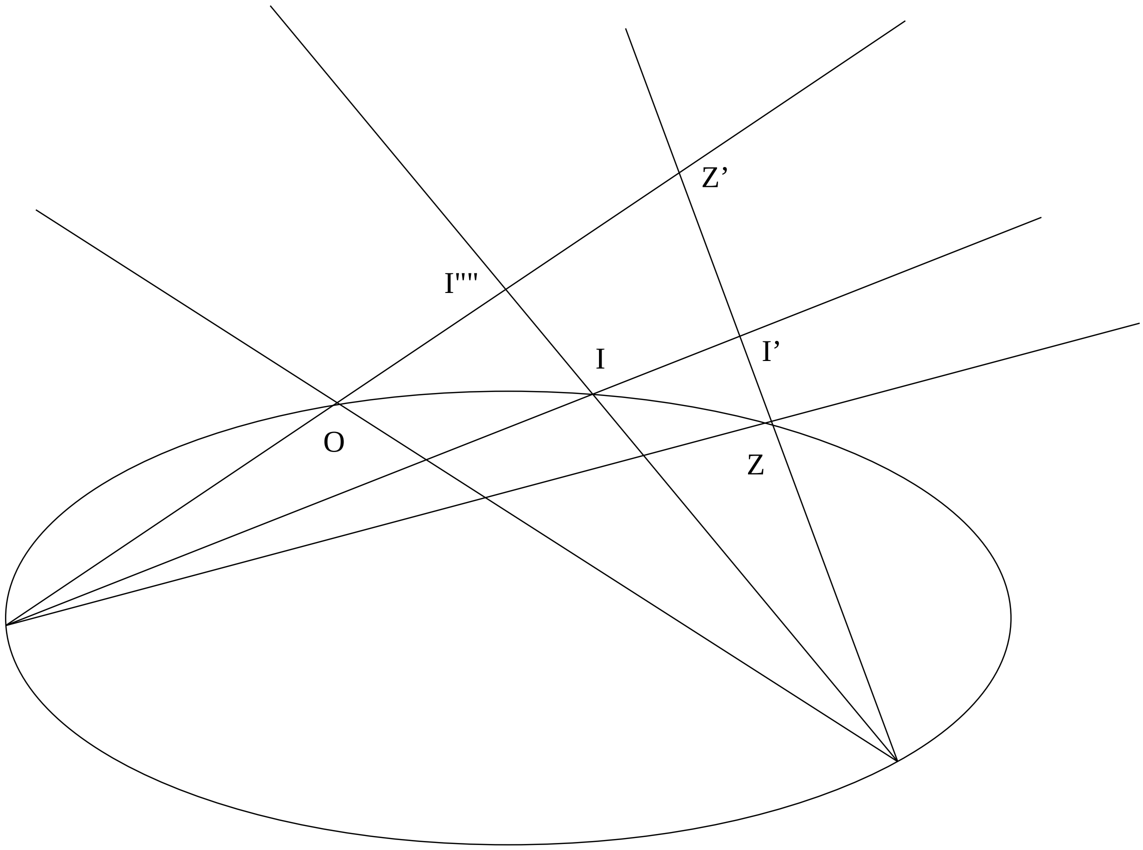
Conic sections are usually introduced as the first curves (as opposed to lines) and are given as the locus of points satisfying an equation of the form

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

We have also studied that the behaviour of this equation is controlled by the following determinants

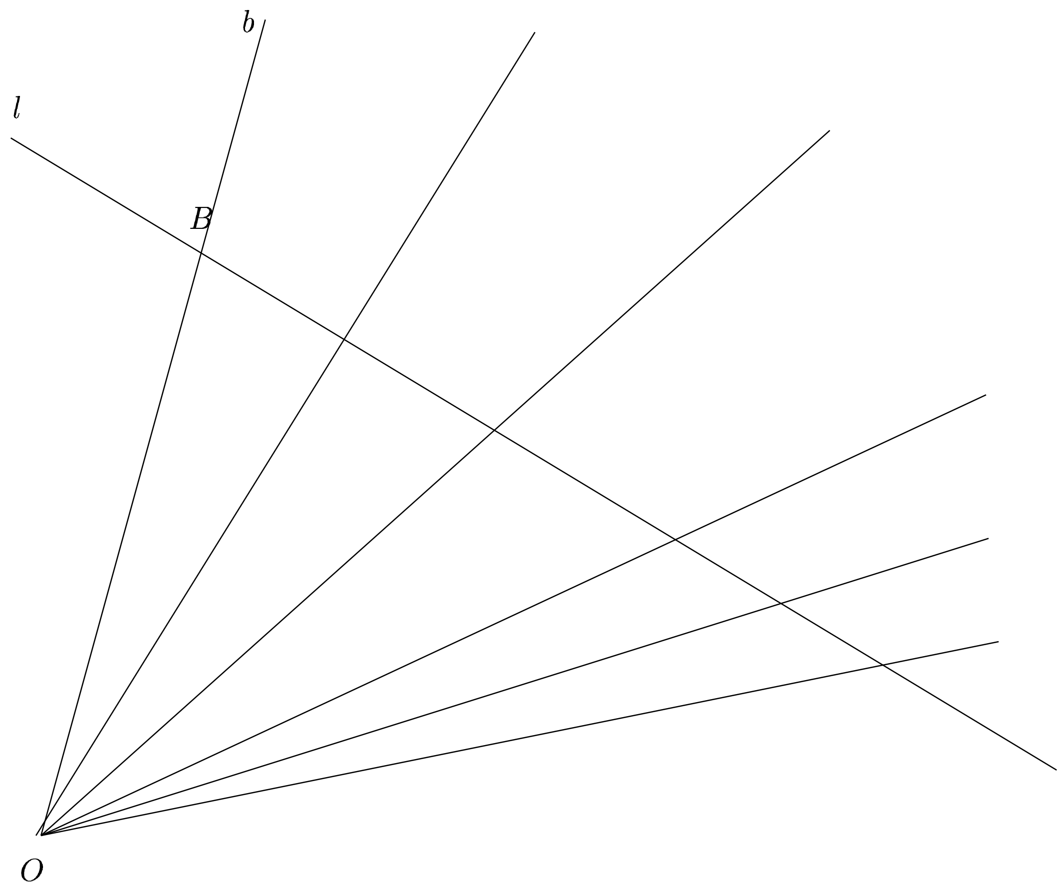
$$D = \det \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \; ; \; d = \det \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

If $D = 0$ then the locus is a pair of lines which are parallel (and could coincide) if $d = 0$. We get curves if $D \neq 0$ which are the hyperbola, parabola and ellipse (or circle) if d is negative, zero or positive (the circle corresponds to the latter when $a = c$). Since we have defined addition and multiplication in geometric terms it should not amaze us to see that the above definition can be made without reference to an algebraic equation.



A conic is the locus of intersection points of pairs of lines that correspond under a projectivity between to pencils

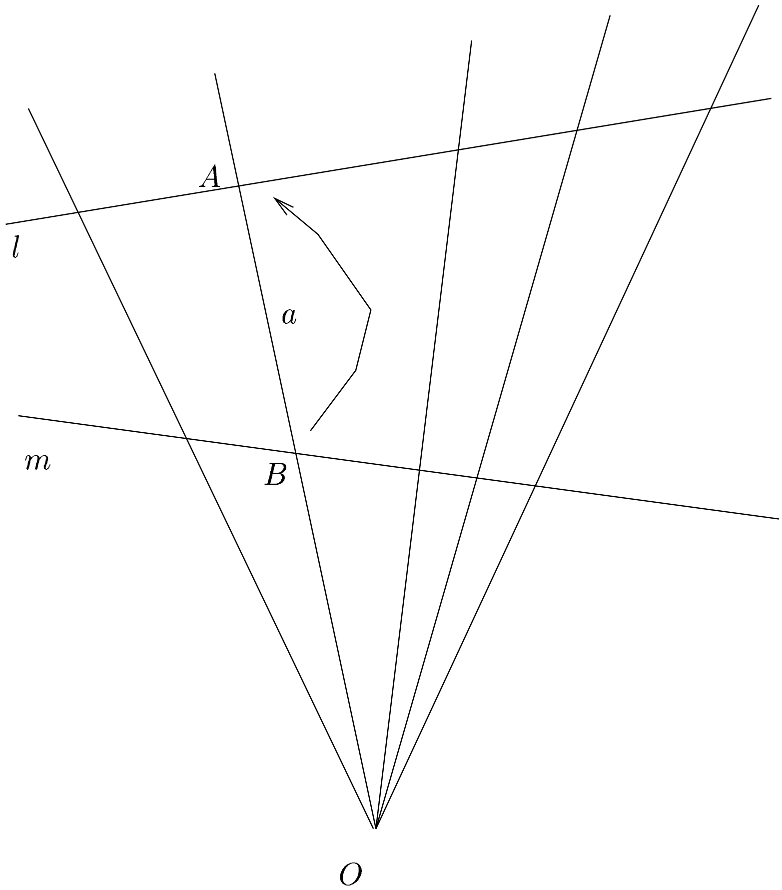
In order to understand this statement let us a fix a projective plane and work within it. The collection of all lines through a fixed point O are in natural 1-1 correspondence with the points on a line l not containing O; any point B on l determines a unique line b joining O and B and conversely, any line b through O meets l in exactly one point B. The locus of lines through a fixed point is called a *pencil*.



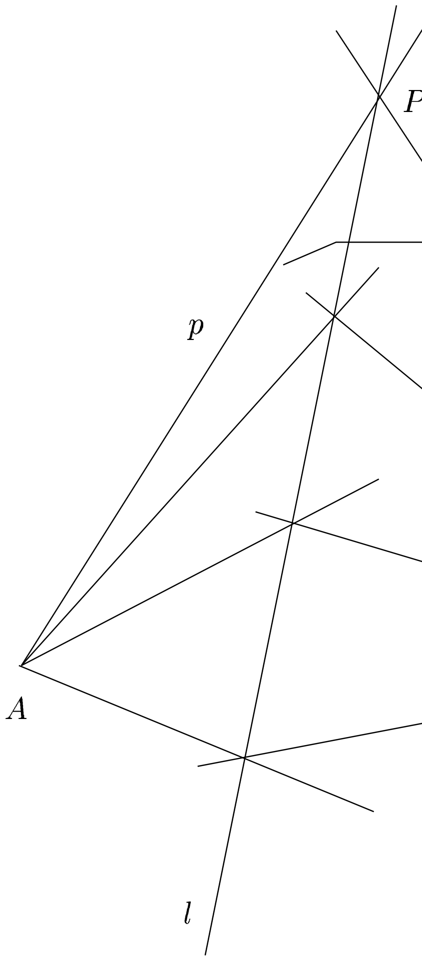
The pencil of lines through a point O is in 1-1 correspondence with the points of l

If l and m are two lines, neither containing a point O, then we have natural correspondence given above between the points of each line and the points of the pencil through O. This sets up a 1-1 correspondence between the two lines. Explicitly, for each point A of l let a be the line joining O and A and let B be the point of m that lies on a. This correspondence is called the *perspectivity* between the points of l and m with center O.

Similarly, if a pair of points A and B are such that neither lies on a line l, then there is a natural 1-1 correspondence between the pencils through A and B, since both the pencils have a 1-1 correspondence with the points of l. Explicitly, if p is a line containing A which meets l in P, we consider the line q joining B with P. This correspondence is called the *perspectivity* between the pencils through A and B with axis l.



Central perspectivity between lines



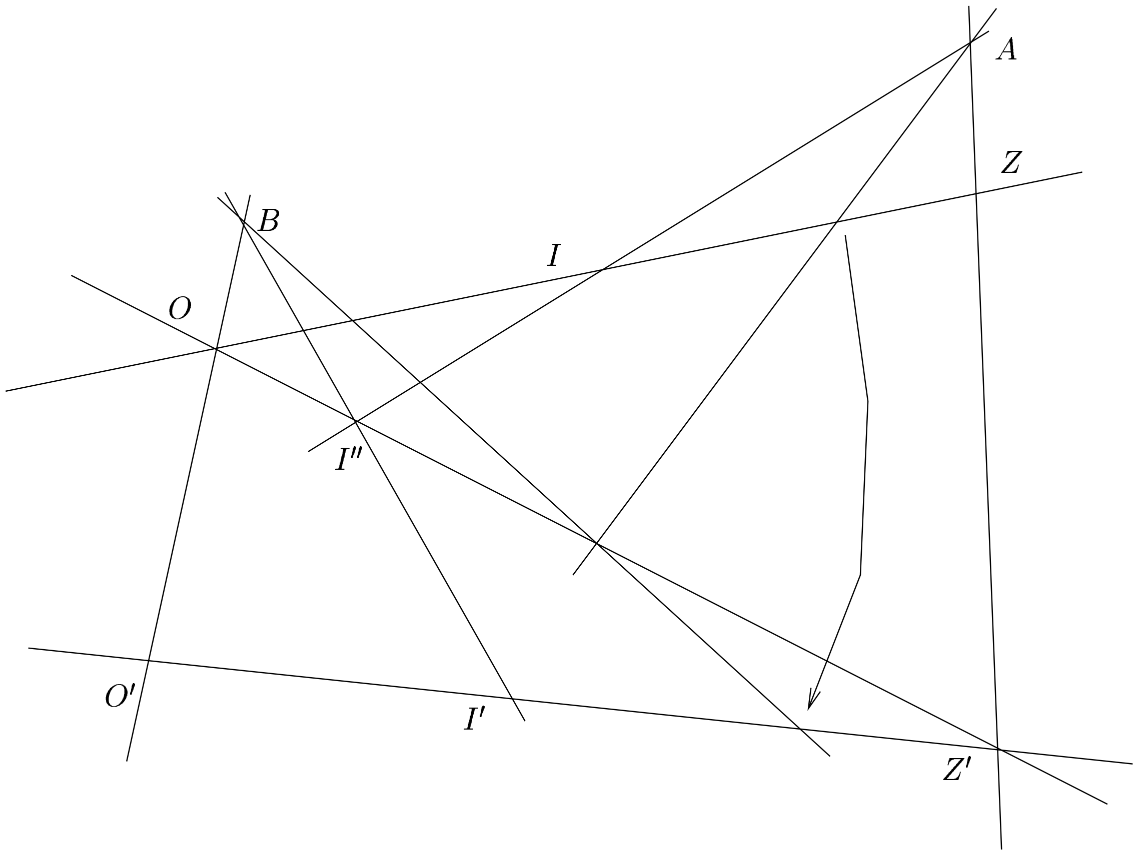
Axial perspecti

In both case a *projective* correspondence or *projectivity* is defined as a composition of perspectivities. Thus, the definition of a conic says, take a pair of points A and B and a projectivity π between the pencils of lines through A and B. Let C be the locus of points of the form $l \cap \pi(l)$ where l is a line through A and $\pi(l)$ the corresponding line through B, then C is a conic.

In order to understand projectivities better we note the following

Exercise 1. Given three points O, I and Z one a line l, and three points O', I', Z' on a line m. Each point A other than the given points on l corresponds to an element $\lambda(A)$ of $K - \{0,1\}$ where K is the underlying field. Similarly, a point B of m other than the given points corresponds to $\mu(B)$ in $K - \{0,1\}$. A perspectivity (and more generally a projectivity) π

can be written as $\mu(\pi(A)) = (a + b\lambda(A))/(c + d\lambda(A))$ where a, b, c and d are in K and $ad - bc \neq 0$.



Central perspectivity from A and then B gives a general projectivity

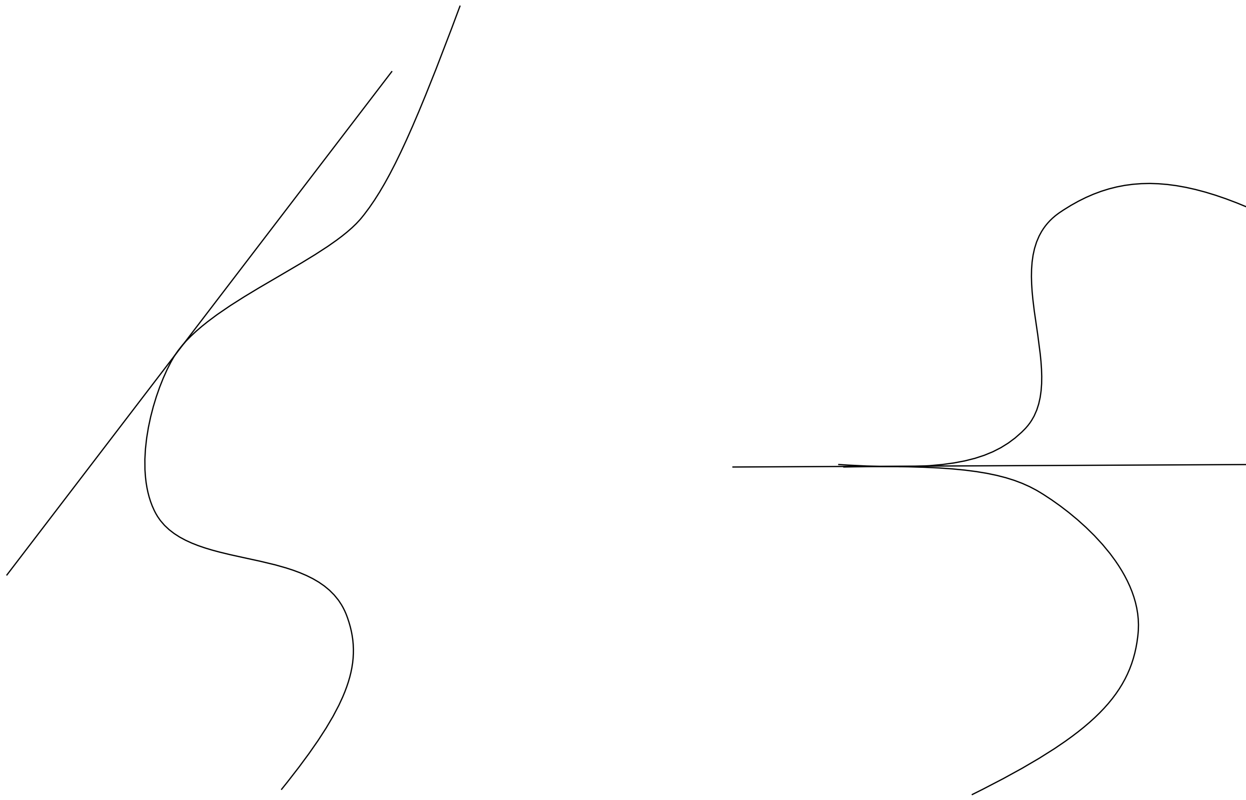
The study of conics in projective geometry is a fascinating one but we leave it here with a pointer to the suggested readings on projective geometry. At the same time we note that it would be rather difficult to study more complicated curves such as the locus of points satisfying $x^3 + y^3 = 1$, using only the incidences between points and lines and not algebra; this is possible *in principle* since the addition and multiplication operations have been defined in terms of the incidence relations. From now on we will use all the familiar notions from algebra and deal with coordinate geometry.

1.2. **Curves.** To emphasise the obvious, the fundamental difference between lines and curves is that the latter are "curved". How do we quantify this "curvy-curvy"-ness or curvature? We shall give a definition and examine some properties.

Let C denote the curve given as the locus of points (x, y) in the coordinate plane which satisfy the equation $f(x, y) = 0$. (Here and elsewhere we will restrict ourselves to polynomial functions since these are the natural outcome of the operations so far defined geometrically). Let $p = (x_0, y_0)$ be a point on the curve. For any line l (for example given by the equation $a(x - x_0) = b(y - y_0)$) that contains this point we can define the *order of contact* between l and C at p as follows. We substitute the parametric solution $(x, y) = (bt + x_0, at + y_0)$ of the line in the equation of C to obtain $F(t) = f(bt + x_0, at + y_0)$. The order of contact $r(l, C, p)$ is then the "largest" power of t that divides $F(t)$. If $F(t) = 0$ then the curve contains the line l and we define $r(l, C, p) = \infty$ otherwise this order is bounded above by the degree of F which is a polynomial in t. Since $t = 0$ corresponds to the point p we always have $r(l, C, p) \geq 1$.

Definition 1. A line l with a higher order of contact with C at p than any other line is called an *osculating line* or tangent to C at p.

Exercise 2. Either $r(l, C, p) \geq 2$ for all l through p or there is a unique line l through p for which $r(l, C, p) \geq 2$. In the former case we say C is *singular* at p and in the latter case we say that C is *non-singular* or *smooth* at p.



Smooth curve and its tangent line

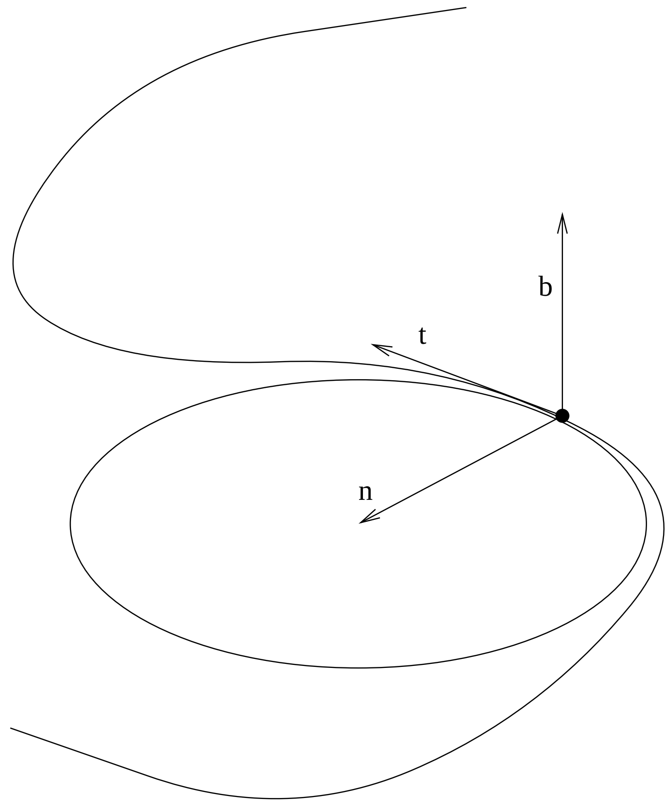
Singular curve and its tangent

Smooth and Singular Curves

In what follows we will concentrate our attention on non-singular curves. The singular case is very interesting but beyond the scope of our current discussion.

In order to motivate one definition of curvature, we first note that a circle is clearly curved. The curvature of a circle should be inversely proportional to the radius since the family of circles touching a line at a given point approach the line as the radius increases. Thus we could define

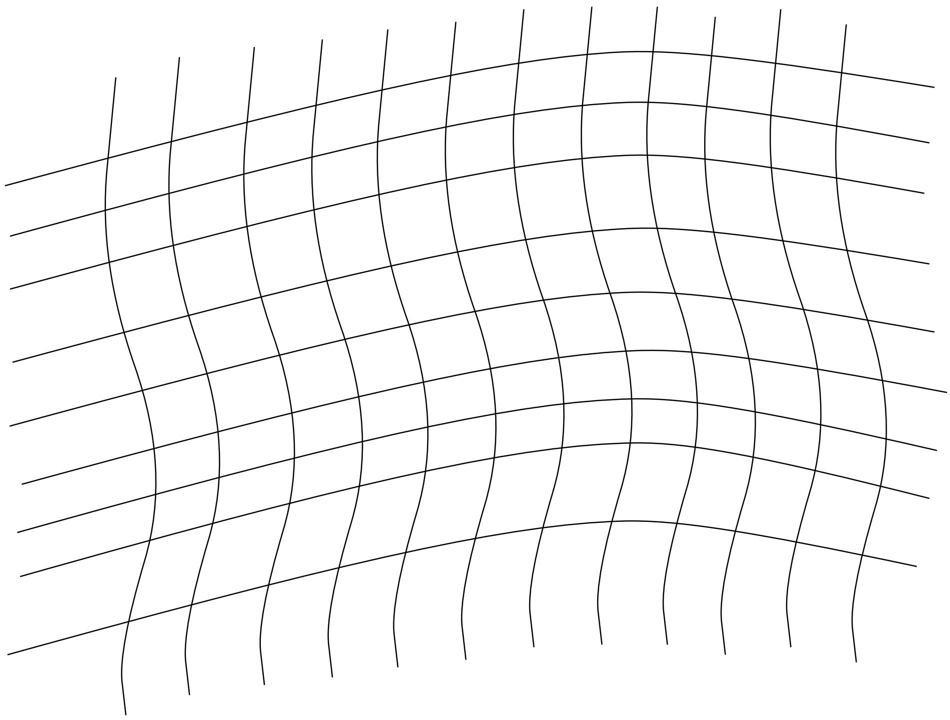
Definition 2. Given a point on a curve the *osculating circle* is the circle that has the highest order of contact with the curve at the given point. The *curvature* of the curve at the given point is the inverse of the radius of this circle.



t: The tangent vector n: The normal vector b:The bi−normal vector

Of course, we must first define the order of contact of a circle and a curve or more generally of two curves, then we must show that a unique circle as defined exists. Note also that once we use circles in a definition we can only make changes of coordinates with preserve circles and their radii; that is, rotations, reflections and translations. In other words, we must fix a notion of distance for curvature to make sense.

Exercise 3. Show that $(u, v) \mapsto (x + y^3, y)$ is a one-to-one and onto correspondence between the (u, v)-plane and the (x, y)-plane. What curves in the (u, v)-plane correspond to lines in the (x, y)-plane? Are these curved? Can we take these to be lines in a geometry?



Curvilinear coordinates

To mimic the definition of the order $r(l, C, p)$ of contact of a line and a curve in order to define $r(C, D, p)$ we need to find a parametric solution of at least one of the curves. Such parametrisations are impossible in general. Instead, we have

Definition 3. We say that $(x(t), y(t))$ is a *parametric solution of* $f(x, y)$ *of order* r if t^r divides $f(x(t), x(t))$.

If C and D are curves defined by $f(x, y) = 0$ and $g(x, y) = 0$ and $p = (x_0, y_0)$ is a point on both the curves then we say that C and D have order of contact (at least) r is there is a *common* parametric solution $(x(t), y(t))$ or order r for both curves so that $(x(0), y(0)) = (x_0, y_0)$. We can define $r(C, D, p)$ as the supremum of such r.

Exercise 4. Show that $r(C, D, p) \geq 2$ for smooth curves C and D if and only if the tangent lines at p to C and D respectively coincide.

In particular, it is always possible to find a circle D which has order of contact 2 at a given point $p = (x_0, y_0)$ on C. In order to find a circle with order of contact 3 or more we need to find conditions so that the following equations have a common solution

$$\begin{array}{lcl} t^3 & | & (x_0 + x_1t + x_2t^2 - a)^2 + (y_0 + y_1t + y_2t^2 - b)^2 - r^2 \\ t^3 & | & f(x_0 + x_1t + x_2t^2, y_0 + y_1t + y_2t^2) \end{array}$$

We collect coefficients of 1, t and t^2 to obtain the following system of equations

$$\begin{array}{rcl} (x_0 - a)^2 + (y_0 - b)^2 - r^2 & = & 0 \\ f(x_0, y_0) & = & 0 \\ (x_0 - a)x_1 + (y_0 - a)y_1 & = & 0 \\ f_x(x_0, y_0)x_1 + f_y(x_0, y_0)y_1 & = & 0 \\ (x_0 - a)x_2 + (y_0 - a)y_2 + x_1^2 + y_1^2 & = & 0 \\ f_{xx}(x_0, y_0)x_1^2 + 2f_{xy}(x_0, y_0)x_1y_1 + f_{yy}(x_0, y_0)y_1^2 + & & \\ f_x(x_0, y_0)x_2 + f_y(x_0, y_0)y_2 & = & 0 \end{array}$$

Here we adopt the subscript notation g_u to denote the (partial) derivative of a polynomial g in the variable u; we note that this derivative is defined formally (without the use of limits) and arises in the above equations due to the binomial expansion which is the Taylor expansion for polynomials.

Exercise 5. A solution for the above equations exists only if

$$r^2 = \frac{(f_x^2 + f_y^2)^3}{(f_{xx}f_y^2 + 2f_{xy}f_xf_y + f_{yy}f_x^2)^2}\Big|_{(x_0,y_0)}$$

Moreover, if we can form square roots then a solution does exist in this case.

Thus we obtain the following formula for the curvature κ of a curve

$$\kappa = \frac{(f_{xx}f_y^2 + 2f_{xy}f_xf_y + f_{yy}f_x^2)}{(f_x^2 + f_y^2)^{3/2}}\Big|_{(x_0,y_0)}$$

Another way to approach curvature is by reversing Newton's law of motion. If a body experiences no acceleration then it must travel along a straight line at constant speed. Thus, we can define the curvature of a curve as (the magnitude of) the acceleration experienced by a body travelling along the curve at constant speed. This definition has the advantage of being applicable to space curves $(x(t), y(t), z(t))$ as well. We again have to surmount a difficulty that it may not be easy to find $(x(t), y(t), z(t))$ so that $(x_t(t), y_t(t), z_t(t))$ is of length 1; certainly there will in general be no polynomial functions that will do the trick. However, we note that acceleration is just $(x_{tt}(0), y_{tt}(0), z_{tt}(0))$ for such a curve so that it will be enough to find a parametric solution of order 3 which has constant speed.

We now apply this method to the curve defined by $f(x, y)$ as above. The above considerations lead to the following pair of conditions

$$\begin{array}{lcl} t^3 & | & f(x_0 + x_1t + x_2t^2, y_0 + y_1t + y_2t^2) \\ t^2 & | & (x_1 + 2x_2t)^2 + (y_1 + 2y_2t)^2 - 1 \end{array}$$

We then have to compute (x_2, y_2) or rather its magnitude. Now we collect coefficients of powers of t as before to obtain the following system of equations

$$\begin{aligned} f(x_0, y_0) &= 0 \\ f_x(x_0, y_0)x_1 + f_x(x_0, y_0)y_1 &= 0 \\ f_{xx}(x_0, y_0)x_1^2 + 2f_{xy}(x_0, y_0)x_1y_1 + f_{yy}(x_0, y_0)y_1^2 + & \\ f_x(x_0, y_0)x_2 + f_y(x_0, y_0)y_2 &= 0 \\ x_1^2 + y_1^2 &= 1 \\ x_1x_2 + y_1y_2 &= 0 \end{aligned}$$

Exercise 6. Solve the above equations to obtain the magnitude $\rho = \sqrt{x_2^2 + y_2^2}$ of the acceleration as

$$\rho = \frac{(f_{xx}f_y^2 + 2f_{xy}f_xf_y + f_{yy}f_x^2)}{(f_x^2 + f_y^2)^{3/2}}|_{(x_0,y_0)}$$

Thus we see that the notion of curvature can indeed be recovered from Newton's law as the acceleration of a particle moving along the curve at constant speed.

We now compute the curvature for any parametric curve $(x(t), y(t), z(t)$ at the point p corresponding to $t = t_0$. To compute curvature following the above definitions we need to re-parametrise the above curve by $t = t_0 + g(s)$ so that the parametric curve $(x'(s), y'(s), z'(s)) = (x(t_0 + g(s)), y(t_0 + g(s)), z(t_0 + g(s)))$ is traversed at constant speed (with respect to the parameter s). The acceleration $(x'_{ss}(0), y'_{ss}(0), z'_{ss}(0))$ gives the curvature of the curve as its magnitude.

Exercise 7. Work with a solution $g(s) = g_1s + g_2s^2$ of order 3 to obtain an expression for the curvature of this curve.

A useful notion is that of the *directional derivative* $D_v(g)(p)$ of a function g at a point $p = (x_0, y_0)$ along a direction $v = (x_1, y_1)$. This is the derivative at $t = 0$ of the function $G(t) = g(x_0 + tx_1, y_0 + ty_1)$; in other words

$$D_v(g)(p) = x_1g_x(x_0, y_0) + y_1g(x_0, y_0) + z_1g(x_0, y_0)$$

We consider the function which assigns to each point of a plane curve C its unit normal, that is

$$(x, y) \mapsto n(x, y) = \frac{(f_x, f_y)}{(f_x^2 + f_y^2)^{1/2}}$$

Exercise 8. Show that the directional derivative $D_v(n)(x, y)$ of $n(x, y)$ is a vector which is orthogonal to $n(x, y)$.

In particular, we note that if v is a non-zero tangent vector to C at a point (x_0, y_0) on C then $D_v(n)(x_0, y_0) = \alpha \cdot v$ for some α.

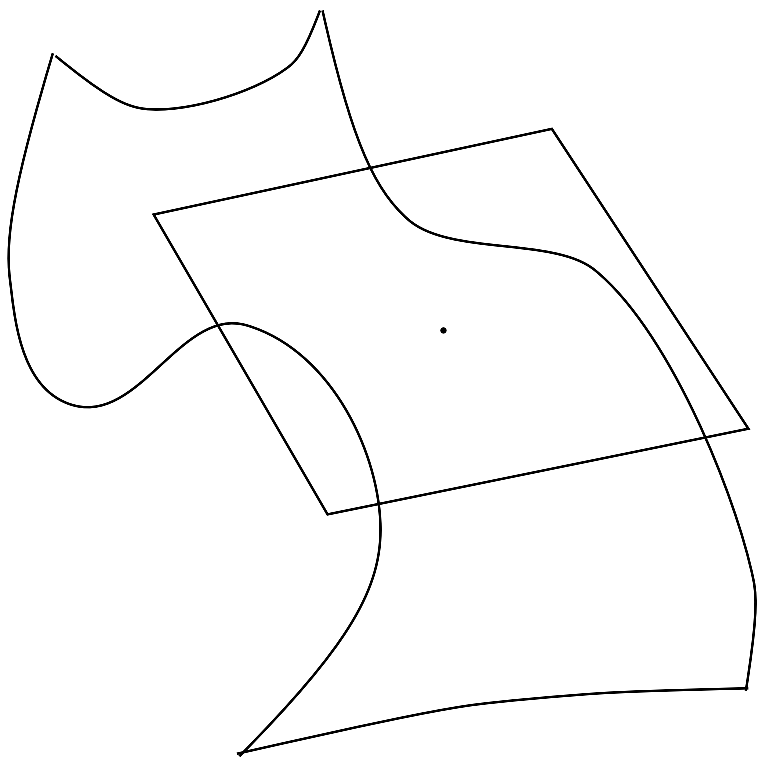
Exercise 9. Show that α is the same as the curvature of C upto sign.

An important aspect of the above calculations is that there is no "intrinsic" curvature to a curve—at least not in a local picture. Any curve can be parametrised in such a way that distance along the curve depends linearly on the parameter; exactly as for a line. The curvature occurs only in the manner in which the curve is embedded in the surrounding plane (or space). We shall see that the behaviour for surfaces is very different.

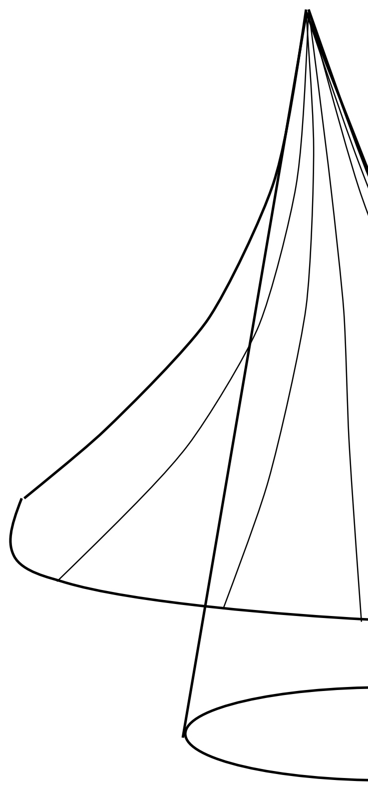
1.3. **Surfaces.** As in the case of curves we will begin with the study of a surface S defined by a single equation $f(x, y, z) = 0$ and eventually treat more general surfaces. If $p = (x_0, y_0, z_0)$ is a point on S, and a line l through p is given in parametric form $(x_0+tx_1, y_0+ty_1, z_0+tz_1)$, then we can define the order of contact $r(l, S, p)$ as the highest power of t which divides the the function $F(t) = f(x_0 + tx_1, y_0 + ty_1, z_0 + tz_1)$. A tangent line is then one which has the largest order of contact among all lines. The locus of all tangent lines describes a conical surface with vertex p. In fact,

Exercise 10. Either $r(l, S, p) \geq 2$ for all lines through p or there is a plane P so that lines l with $r(l, S, p) \geq 2$ are precisely lines lying in the plane and passing through p. The latter case occurs if and only if the vector $(f_x(p), f_y(p), f_z(p))$ is non-zero. By a replacing the equation $f = 0$ by a non-zero multiple we can ensure that the vector $(f_x(p), f_y(p), f_z(p))$ is a unit vector, which is then called the *unit normal* to the surface.

As in the case of curves we call the former case the *singular* surfaces and the latter are called *smooth* or non-singular surfaces. We will restrict our attention to smooth surfaces.



Smooth surface and its plane of tangent lines



Singular surface and it

We define the linear space of tangent vectors $T_p(S)$ to S at p as the space of all vectors v so that the parametric line $p + tv$ is tangent to S at p.

Exercise 11. Show that the tangent space is precisely,

$$\{v = (x_1, y_1, z_1) \mid f_x(x_0, y_0, z_0)x_1 + f_y(x_0, y_0, z_0)y_1 + f_z(x_0, y_0, z_0)z_1\}$$

In other words v is tangent if and only if $D_v(f)(p) = 0$.

The curvature of curves defined in the previous section can be formulated as follows. We have a map from a plane curve C to the unit circle given by sending each point to the unit tangent vector at that point (we need to choose a "direction" along the curve but that can be done locally). As we saw the derivative of this map is the curvature. Now the unit normal direction is a right-angle rotation from the tangent vector, so we could equally well have used the map which takes a point to its unit normal. The latter map also makes sense for a surface S in 3-space; we have a natural map

$$p \mapsto n(p) = \frac{(f_x(p), f_y(p), f_z(p))}{(f_x(p)^2 + f_y(p)^2 + f_z(p)^2)^{1/2}}$$

which sends S to the (unit) sphere. Curvature should be a measure of the derivative of this map. Let v be any vector in $T_p(S)$ or equivalently v be a vector orthogonal to $n(p)$.

Exercise 12. Show that $D_v(n)(p)$ is orthogonal to $n(p)$ also. (Hint: Use the equation $n \cdot n = 1$ for all p).

Thus, if v and w are linearly independent vectors in $T_p(S)$ (and hence form a basis of this vector space), then we have

$$D_v(n)(p) = av + bw \text{ and } D_w(n)(p) = cv + bw$$

for some constants a, b, c and d.

Definition 4. We define the Gaussian curvature of S at p to be the determinant $ad - bc$ of the above linear transformation.

Exercise 13. Show that for any function f, vectors v, w and constant a we have $D_{(av+w)}(f) = aD_v(f) + D_w(f)$. Now show that the above definition is independent of the choice of vectors v and w.

We can additionally justify the above definition by noting that the Gaussian curvature of a plane is 0. In order to understand the Gaussian curvature better we must first understand the notion of "straight lines" or *geodesics*. To begin with let us examine curves lying on the surface. Let $p = (x_0, y_0, z_0)$ be a point of S and $(x(t), y(t), z(t))$ be a curve on S passing through p at $t = 0$.

Exercise 14. Show that the tangent vector $(x_t(0), y_t(0), z_t(0))$ is in the tangent space $T_p(S)$.

We have already seen that a curve has non-trivial curvature if the acceleration required to travel along it at constant speed is non-zero. Thus a curve must be considered *straight* on the surface S if the *projection* of this acceleration into $T_p(S)$ is zero. In other words:

Definition 5. Let $(x(t), y(t), z(t))$ be a parametrised curve (of some order $r \geq 2$) on the surface S. Moreover, suppose that the speed $((x_t)^2 + (y_t)^2 + (z_t)^2)^{1/2}$ is a constant. We say that the curve is *geodesic* on S if the acceleration at any point of the curve is a multiple of the unit normal to S at that point.

Exercise 15. Consider a parametric solution of order 3 to the equations $f(x(t), y(t), z(t)) = 0$ and $(x_t)^2 + (y_t)^2 + (z_t)^2 = c$; morover suppose that the acceleration is a multiple of the normal to S. Show that we obtain the equation for the acceleration vector

$$(x_2, y_2, z_2) = -\frac{Q_f(p)(x_1, y_1, z_1)}{f_x(p)^2 + f_y(p)^2 + f_z(p)^2}(f_x(p), f_y(p), f_z(p))$$

where, $p = (x_0, y_0, z_0)$ is the "starting point", (x_1, y_1, z_1) is the initial tangent vector and Q_f is the quadratic form given by the second derivatives of f, that is

$$Q_f(a,b,c) = f_{xx}a^2 + 2f_{xy}ab + 2f_{xz}ac + f_{yy}b^2 + 2f_{yz}bc + f_{zz}c^2$$

The above equation can be written as

$$(x_{tt}, y_{tt}, z_{tt})(p) = -\frac{Q_f(x_t, y_t, z_t)(p)}{f_x(p)^2 + f_y(p)^2 + f_z(p)^2}(f_x(p), f_y(p), f_z(p))$$

Given any point (x_0, y_0, z_0) on the surface and a tangent direction (x_1, y_1, z_1) to S at that point, we can inductively solve this equation to obtain the terms upto any required order and obtain the geodesic in parametric form.

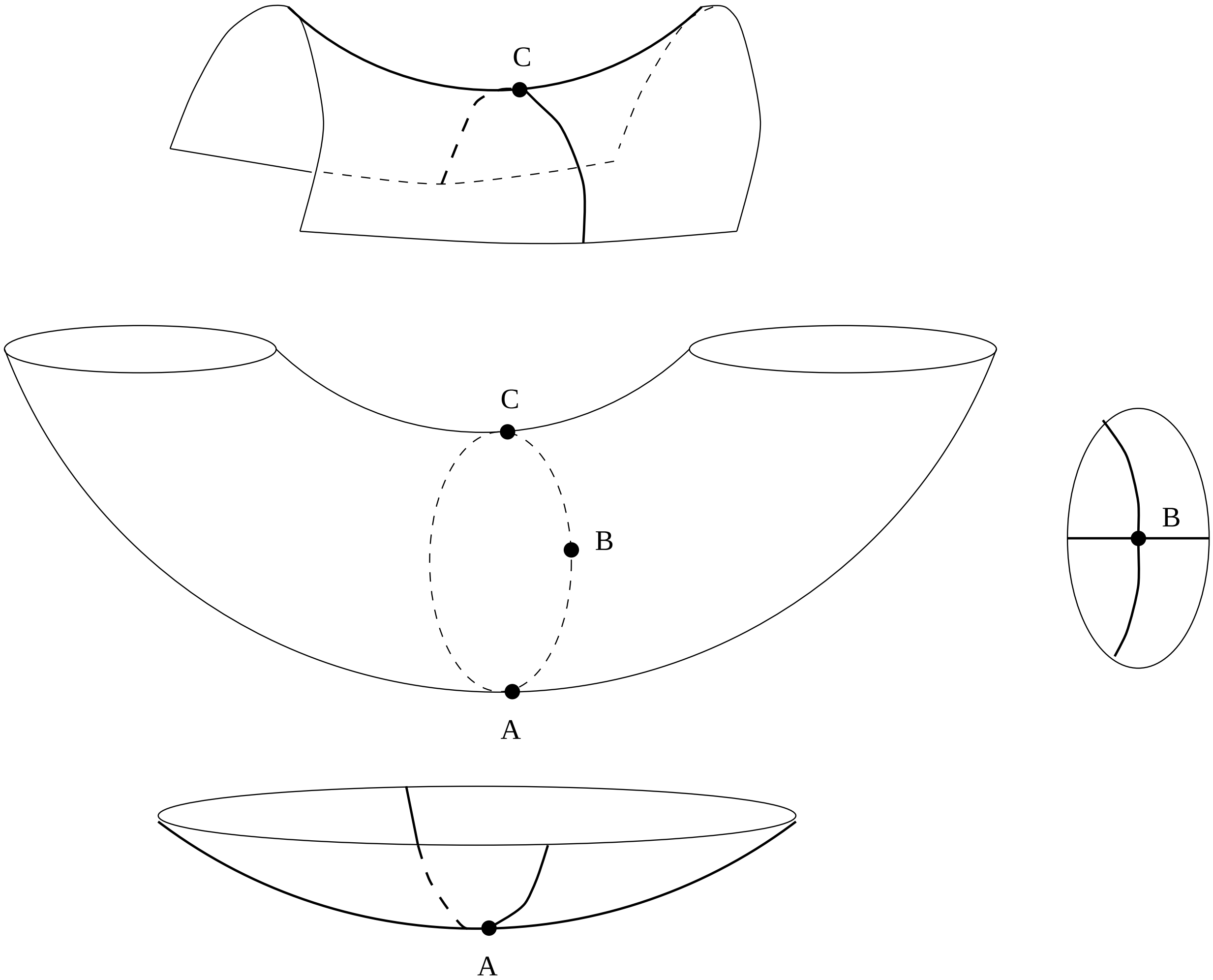
Consider the quadratic form Q obtained by restricting the form $Q_f/(f_x^2 + f_y^2 + f_z^2)^{1/2}$ to the tangent space $T_p(S)$.

Theorem 1. *Let Q be a quadratic form on a vector space V (over the real numbers) there is an orthonormal basis $\{e_1, \ldots, e_n\}$ of V so that Q can be written as*

$$Q(u_1e_1 + \cdots + u_ne_n) = a_1u_1^2 + \ldots a_ne_n^2$$

In particular, we have an orthonormal basis $\{e_1, e_2\}$ for the tangent space $T_p(S)$ so that $Q(ue_1 + ve_2) = ku^2 + lv^2$. Euler called the numbers k and l the principal curvatures of S.

Exercise 16. Show that the Gaussian curvature of S is kl.



The thick lines denote the geodesics of extremal curvature

Examples of different curvatures on a tube

We shall now show how one can use geodesics to compute curvature. Let p be any point on the surface S, then for each tangent direction w in $T_p(S)$ we can form the parametric geodesic $g_{p,w}(t)$ of some order r. Let e_1, e_2 be a basis of $T_p(S)$ as above and $w = ue_1 + ve_2$. Consider the map

$$(u, v) \mapsto p(u, v) = g_{p,ue_1+ve_2}(1)$$

which gives a parametric form for S upto order r. This parametric form is called the geodesic normal form or geodesic normal co-ordinates and is determined by the geometry of the surface (that is, by understanding the equations of "straight lines" or geodesics on the surface).

To simplify things we take p to be the origin (by translation) and let e_3 to be the unit normal to S at p. We use $\{e_1, e_2, e_3\}$ as a basis in which to express points of space. The above map then becomes (for $r = 4$),

$$p(u, v) = (u + A(u, v), v + B(u, v), -frac12Q(u, v) + C(u, v))$$

where A, B and C are homogeneous of degree 3.

Exercise 17. By using the geodesic equation show that $A = Q(u, v)Q_u(u, v)/6$ and $B = Q(u, v)Q_v(u, v)/6$.

By definition of $p(u, v)$ the lines through the origin in the (u, v) plane go to geodesics in S. What about other lines? Consider the line $l = (u_0 + su_1, v_0 + sv_1)$ parametrised by s in the (u, v) plane. It becomes a curve $\gamma(s) = p(u_0 + su_1, v_0 + sv_1)$ in space. The velocity vector of this curve is $\gamma_s(s) = p_u u_1 + p_v v_1$ and the acceleration is

$$a(s) = \gamma_{ss}(s) = p_{uu}u_1^2 + 2p_{uv}u_1v_1 + p_{vv}v_1^2$$

We need to find the component of this accleration in the tangent plane to S, i. e. is the space of p_u and p_v.

Exercise 18. Prove the following formulas valid modulo order 2 in u_0 and v_0

$$\begin{aligned} a(s) \cdot p_u &= kl/3v_1(v_0u_1 - u_0v_1) \\ a(s) \cdot p_v &= kl/3u_1(v_0u_1 - u_0v_1) \end{aligned}$$

One checks that p_u and p_v are unit vectors modulo order 2 in u_0 and v_0. Thus, the magnitude of the acceleration is $(kl/3)(v_0u_1 - u_0v_1)$ times the length of the vector (u_1, v_1) along the line l in the (u, v) plane. Thus the curvature of the image of lines parallel to the origin in the (u, v) plane is also related to the Gaussian curvature of S; which thus also measures the deviation from the parallel postulate.

1.4. **Manifolds.** We can try to generalise the above results from two variables u, v to n variables $u_1, \ldots, u_n$ for any positive integer n. A manifold M in parametric form is the locus of points (in some $\mathbb{R}^k$) obtained as the image of a vector-valued function $x(u_1, \ldots, u_n)$. Moreover, to ensure that we have n "directions of freedom" we also require the derivatives $x_{u_1}, \ldots,$ x_{u_n} to be a linearly independent set at every point; we use $T_p(M)$ to denote the linear span of these vectors which is called the tangent space to M at p. (For those who know more, this is a local manifold and not a global one).

If $(u_1(t), \ldots, u_n(t))$ is an n-tuple of functions of one variable we obtain a curve in M. The velocity and acceleration of this curve as a curve in $\mathbb{R}^k$ are

$$\begin{aligned} v &= x_{u_1}u_{1,t} + \cdots + x_{u_n}u_{n,t} \text{ and} \\ a &= \sum_{i,j} x_{u_iu_j}u_{i,t}u_{j,t} \sum_i x_{u_i}u_{i,tt} \end{aligned}$$

The vector v is in the tangent space to M. We need to understand the projection of acceleration into the tangent space to see whether a curve is indeed "curved" in M or is a geodesic.

Exercise 19. Let $v_1, \ldots, v_n$ be linearly independent vectors. Show that the matrix whose ij-th entry is $g_{ij} = v_i \cdot v_j$ is a positive definite symmetric matrix. Let h_{ij} denote the matrix entries of its inverse. The orthogonal projection of a vector a to the linear span of $v_1, \ldots, v_n$ is given by $\sum_{i,j} h_{ij}(a \cdot v_i)v_j$.

Thus we see that the orthogonal projection of the acceleration into the tangent space of M is given by

$$\sum_{i,j,k,l} (h_{kl}(x_{u_i u_j} \cdot x_{u_k})u_{i,t}u_{j,t})x_{u_l} + \sum_i u_{i,tt}x_{u_i}$$

Where we will use $g_{ij} = (x_{u_i} \cdot x_{u_j})$ and h_{ij} for the matrix entries of the inverse of the matrix (g_{ij}). Now we have the following equations for the derivatives of g_{ij}

$$(g_{ij})_{u_k} = (x_{u_i u_k} \cdot x_{u_j}) + (x_{u_i} \cdot x_{u_j u_k})$$

We "solve" these to obtain

$$(x_{u_i u_j} \cdot x_{u_k}) = (1/2)((g_{ik})_{u_j} + (g_{jk})_{u_i} - (g_{ij})_{u_k})$$

Finally, we can put

$$\Gamma_{ij,l} = \sum_k h_{kl}(1/2)((g_{ik})_{u_j} + (g_{jk})_{u_i} - (g_{ij})_{u_k})$$

so that the components of acceleration along x_{u_k} are given by $\sum_{i,j,k} \Gamma_{ij,k}u_{i,t}u_{j,t} + u_{k,tt}$.

First of all we note that the $\Gamma_{ij,k}$ and hence the acceleration of a curve on M can be computed one we are given the g_{ij} as functions of u_i. We no longer need the "crutch" of the vector-valued functions $x(u_i)$. Thus, the geometry of the manifold M is *entirely* described by the matrix-valued function (g_{ij}) with values in positive-definite matrices. This function gives the speed of a curve $(u_i(t))$ as $\sum g_{ij}u_{i,t}u_{j,t}$ and the accleration is given by the formula above.

Exercise 20. Use the geodesic normal form of the surface as given above and compute the speed of a curve $(u(t), v(t))$ to be

$$u_t^2 + v_t^2 + (kl/3)(uv_t - vu_t)^2$$

modulo order 3 terms in u and v.

Thus the curvature can also be obtained by comparing the speed of curves with the "usual" speed in the (u, v) plane. We shall see that this is the key idea used by Riemann to understand curvature in many dimensions.

Secondly, we note that the equation for a curve on M with 0 accleration can easily be solved *inductively* given the starting point and the velocity at that point. In other words given $(u_{1,0}, \ldots, u_{n,0})$ and $(u_{1,1}, \ldots, u_{n,1})$ we consider a curve given by the functions $u_i(t) = u_{i,0} + u_{i,1}t + \ldots$. By comparing the coefficients of t^r in the equation $\sum_{i,j} \Gamma_{ij,k}u_{i,t}u_{j,t} + u_{k,tt} = 0$, we can inductively solve for $u_{i,2}$, $u_{i,3}$ and so on.

Let p be a fixed point on M and choose a collection of vectors e_1, ... e_n so that (g_{ij}) becomes the identity matrix with respect to this basis. For each n-tuple $(v_1, \ldots, v_n)$ we consider the geodesic $\gamma_{(v_1,\ldots,v_n)}(t)$ (upto some order in t) obtained as above, starting at p and proceeding along $\sum_i v_i e_i$. One easily sees that the functions

$$u_i(v_1, \ldots, v_n) = u_i(\gamma_{(v_1,\ldots,v_n)}(1))$$

give another parametric form (in the variables v_1, ... ,v_n) for the manfiold M (upto some order r). This form is called the geodesic normal form for M. We can re-express any function on M in terms of the new variables by substitution. Similarly, we can compute the speed of curves $(v_1(t), \ldots, v_n(t)$ in terms of a new matrix (G_{ij}) which can easily be computed in terms of the g_{ij} and the expression of u's as functions of v's.

It is clear that this special parametric form should reflect the geometry of M. In fact, Riemman proved

Theorem 2. *If the manifold M is expressed in the variables $(v_1, \ldots v_n)$ in geodesic normal form, and the expression for spped of a curve takes the form*

$$\sum_{ij} G_{ij} v_{i,t} v_{j,t} = \sum_i v_{i,t}^2 + \sum_{i<j,k<l} c_{ij,kl}(v_i v_{j,t} - v_j v_{i,t})(v_k v_{l,t} - v_l v_{k,t})$$

modulo order 3 terms in the variables v_i.

The second term can be thought of as a quadratic form $\sum_{i<j,k<l} c_{ij,kl} z_{ij} z_{kl}$ in a new set of variables z_{ij}. By diagonalising this quadratic form as earlier we obtain $n(n-1)/2$ "eigenvalues" which are called the *principal sectional curvatures* of M in analogy with the two dimensional case given in the exercise above. Many interesting properties of curvature have been developed over the years. Riemann himself showed that the curvature is 0 at all points of M if and only if M is Euclidean. He also claimed that the functions on M given by the principal sectional curvatures uniquely determine the geometry of M. This claim has only been proved partially.

1.5. **Suggested Reading.** Most "advanced" books on Differential Geometry or on Advanced Calculus will have a good portion of this material. Here are some books that I have found useful.

1. Hicks, *Differential Geometry.*
2. Spivak, *Calculus on Manifolds.*